DOCUMENT RESUME

ED 143 513 SE 022 990

AUTHOR	Ayre, H. Glenn; And Others
TITLE	Analytic Geometry. Teacher's Commentary, Unit No. 65. Revised Edition.
INSTITUTION	Stanford Univ., Calif. School Mathematics Study Group.
SPONS AGENCY	National Science Foundation, Washington, D.C.
PUB DATE	65
NOTE	490p.; For related Student Text, see SE 022 989; Not available in hard copy due to marginal legibility of original document
EDRS PRICE	MF-$1.00 Plus Postage. HC Not Available from EDRS.
DESCRIPTORS	Algebra; *Analytic Geometry; Geometric Concepts; *Instructional Materials; Mathematics Education; *Secondary Education; *Secondary School Mathematics; *Teaching Guides
IDENTIFIERS	*School Mathematics Study Group

ABSTRACT

This is the teacher's guide to the SMSG text ANALYTIC GEOMETRY. The text is designed to be used as a one-semester course for 12th grade students. Included in this guide are: (1) suggested length of study for each chapter; (2) discussion of each chapter that is in the student text; (3) comments keyed to the pages of the student's text to provide explanation and background for the teacher; (4) answers to exercises; and (5) discussion of supplementary materials in the text. (RH)

****
* Documents acquired by ERIC include many informal unpublished *
* materials not available from other sources. ERIC makes every effort *
* to obtain the best copy available. Nevertheless, items of marginal *
* reproducibility are often encountered and this affects the quality *
* of the microfiche and hardcopy reproductions ERIC makes available *
* via the ERIC Document Reproduction Service (EDRS). EDRS is not *
* responsible for the quality of the original document. Reproductions *
* supplied by EDRS are the best that can be made from the original. *
****

ED143513

TEACHER'S COMMENTARY UNIT NO. 65

SMSG

ANALYTIC GEOMETRY

SE 022 990

SCHOOL MATHEMATICS STUDY GROUP

Analytic Geometry

Teacher's Commentary

REVISED EDITION

Prepared by:

H. Glenn Ayre	Western Illinois University
William E. Briggs	University of Colorado
Daniel Comiskey	The Taft School, Watertown, Connecticut
John Dyer-Bennet	Carleton College, Northfield, Minnesota
Daniel J. Ewy	Fresno State College
Sandra Forsythe	Cubberley High School, Palo Alto, California
James H. Hood	San Jose High School, San Jose, California
Max Kramer	San Jose State College, San Jose, California
Carol V. McCamman	Coolidge High School, Washington, D.C.
William Wernick	Evander Childs High School, New York, New York

Stanford, California
Distributed for the School Mathematics Study Group
by A. C. Vroman, Inc., 367 Pasadena Avenue, Pasadena, California

Financial support for School Mathematics
Study Group has been provided by the
National Science Foundation.

Permission to make verbatim use of material in this book must be secured from the Director of SMSG. Such permission will be granted except in unusual circumstances. Publications incorporating SMSG materials must include both an acknowledgment of the SMSG copyright (Yale University or Stanford University, as the case may be) and a disclaimer of SMSG endorsement. Exclusive license will not be granted save in exceptional circumstances, and then only by specific action of the Advisory Board of SMSG.

© 1965 by The Board of Trustees
of the Leland Stanford Junior University.
All rights reserved.
Printed in the United States of America.

ANALYTIC GEOMETRY

Teacher's Commentary

Contents

INTRODUCTION 1

A WORD ABOUT THE EXERCISES 2

A SUGGESTED TIME SCHEDULE 3

Chapter

1. ANALYTIC GEOMETRY 5
2. COORDINATES AND THE LINE 7
3. VECTORS AND THEIR APPLICATIONS 83
4. PROOFS BY ANALYTIC METHODS 123
5. GRAPHS AND THEIR EQUATIONS 147
6. CURVE SKETCHING AND LOCUS PROBLEMS 191
7. CONIC SECTIONS 269
8. THE LINE AND THE PLANE IN 3-SPACE 313
9. QUADRIC SURFACES 347
10. GEOMETRIC TRANSFORMATION 425

Supplement to Chapter 2 449

Supplement D (Supplement to Chapters 2, 3, and 8) 463

Supplement to Chapter 7 475

Supplement to Chapter 10 485

INTRODUCTION

The text Analytic Geometry had its beginnings in 1962 when a small committee of mathematicians and teachers met to discuss the question as to whether there was a need for a new text in analytic geometry for high school, and whether the School Mathematics Study Group should undertake to write one. Since the conclusion was affirmative, some guidelines were prepared to indicate the form and content desired.

In the summer of 1963 an experimental text and accompanying commentary were prepared by an SMSG writing team consisting of university mathematicians and high school teachers. During the following school year this text was used by about 30 teachers in schools distributed from California to New England, but mostly in 2 centers where the teachers had the benefit of conferences with each other and with an interested college professor. The complete revision of the text and commentary in the summer of 1964 took into account both the comments and criticisms of these teachers, and the recommendations of an advisory committee of the SMSG Board. We are deeply indebted to those who helped with suggestions, especially to the teachers who used the experimental text.

Analytic Geometry is intended for use as a one-semester course in the 12th grade. It is expected that the students would have completed SMSG Intermediate Mathematics or the equivalent. If it is planned to use Elementary Functions with the same class, it is suggested that that text be used before the Analytic Geometry. However, knowledge of Elementary Functions has not been assumed in this text.

The suggested time schedule here is only tentative; the teacher will adapt it to the particular class. Certain topics are presented here for completeness; for example, some of the work on forms of an equation of a line, on conic sections, or on vectors, will have been studied previously by many classes. Very little time need be spent on familiar work, giving more time for new topics or for supplementary work.

We believe that a reasonably well-prepared class of the students who elect 12th grade mathematics can complete our basic text (Chapters 1 to 10) in a semester. The material in the supplementary chapters was placed there because it was not felt essential to the continuity of the course. However, we feel that this is important and interesting material; we think that it is within the grasp of able students and will broaden their mathematical background.

It is hoped that good classes and individual able students will use the supplementary chapters.

Following the opening remarks for each chapter in this Commentary, you will find running comments keyed in the margin to the pages of the student's text. These contain further explanation and background which we hope will be useful to you.

A WORD ABOUT THE EXERCISES

Some of the exercises are designed to provide just exercise, but you will find that some others are far from routine. Within each set of exercises the arrangement is usually from the more routine to the more complex problems. The most difficult problems are listed separately as "Challenge Problems". A few problems have been included which extend the material beyond the regular textual treatment. We advise you to look at each such problem before assigning it to a student so that you may ascertain whether it is appropriate and how much time it will consume.

We cannot suggest appropriate class assignments since they will vary with the preparation and ability of the class. Of course, enough drill work should be included to fix the fundamental skills and concepts. In the case of a well-prepared class, the drill-type problems might be omitted entirely on any topic previously studied. While the particular problems assigned will vary with the class and perhaps even with the individual pupils, it is hoped that all students will be assigned some of the problems which may be more time consuming but which will show them some of the "fun" of Analytic Geometry.

Solutions for the exercises appear at the point in the running commentary corresponding to the placement of the problems in the student's text. Any given problem may have several acceptable solutions; therefore, the solution presented here should not be considered as the "right", or only, solution. The student is encouraged frequently to use his own judgment in pursuing a solution; hence, if he presents a solution which is correct, it should be accepted.

A SUGGESTED TIME SCHEDULE

The basic text (Chapters 1 to 10) was designed to be covered in one semester of eighteen weeks. The time schedule given below is the result of combining the opinions of the authors with the experience of the teachers who used the preliminary edition.

If you find that your class is falling behind the suggested schedule, you may wish to compensate by treating some topics in less depth or by assigning fewer exercises. If this procedure is not satisfactory, you probably should consider cutting short, first on Chapter 10 and then on Chapter 9. The text was designed so that the least loss to the students would occur in this circumstance.

Chapter	No. of Days	Cummulative Total
1. Analytic Geometry	1	1
2. Coordinates and the Line	10	11
3. Vectors and Their Applications	12	23
4. Proofs by Analytic Methods	8	31
5. Graphs and Their Equations	9	40
6. Curve Sketching and Locus Problems	11	51
7. Conic Sections	9	60
8. The Line and the Plane in 3-space	7	67
9. Quadric Surfaces	10	77
10. Geometric Transformations	8	85

Chapter 1

ANALYTIC GEOMETRY

Chapter 1 is a brief introduction to the text. It is intended to give the students an idea of what analytic geometry is and to show them they already know something about the subject. If possible, they should read it before the first meeting of the class and reread it at intervals during the course.

Since coordinate systems are so important in analytic geometry, it is advisable to discuss in class some of the examples mentioned. The students should be asked to explain latitude and longitude, which are mentioned but not defined in the text. They might be invited to suggest other coordinate systems for a line, a plane, space, a spherical surface, and a torus. However, the coordinate systems which are important in the course are treated in detail later, so not much class time should be spent on them at this point.

Chapter 1 also includes a discussion of the reasons for studying analytic geometry. It is felt that students should know something of the role of analytic geometry among the various branches of mathematics, and that they should realize that their main goal is not information about the particular topics studied, but rather understanding of and ability to use the techniques of analytic geometry.

Analytic Geometry really began when it was realized that every geometric object and every geometric operation can be referred to the number system and, hence, to algebra. The most significant steps in this arithmetization of geometry were taken by two French mathematicians, Pierre Fermat (1601 - 1655) and René Descartes (1596 - 1650). Fermat began work on analytic geometry in 1629 but his treatise Ad Locus Planos et Solidos Isagoge was not published until 1679. Chief credit, therefore, is given to Descartes whose Geometrie was published in 1637 and who influenced the work of many mathematicians. In the Geometric, one finds the earliest unification of algebra and geometry. Apollonius and other Greek mathematicians had used coordinates to locate points in a geometric figure. It was Descartes who introduced the algebraic representation of a curve or surface by an equation involving two or three variables.

Descartes' book does not contain a systematic development of the subject such as you find in this text. The method must be constructed from isolated statements in different parts of the treatise. It is interesting that Fermat's work included the equations $y = mx$, $xy = k$, $x^2 + y^2 = a^2$, $x^2 \pm a^2y^2 = b^2$ for lines and conics.

Many mathematicians extended Descartes' work. Among these were John Wallis in his *Tractatus de Sectionibus Conics* and John DeWitt in his *Elementa Curvarum Linearum*. Most of the work of Descartes and his contemporaries was concerned with the geometry of Apollonius. Newton worked with algebraic equations in his study of cubic curves in 1703. The first analytic geometry of conic sections divorced from the work of Appollonius was developed by Euler in his *Introductio* in 1748.

Since that time the methods of Analytic Geometry have become the most significant in the study of geometry. In more advanced mathematics they have essentially replaced the synthetic method. More recently vector methods have been incorporated in Analytic Geometry and are being used more and more widely in mathematical applications.

Teacher's Commentary

Chapter 2

COORDINATES AND THE LINE

This chapter is fundamental to the rest of the book. In it we discuss coordinate systems for a line and a plane. We also treat the analytic geometry of lines in a plane. A good deal of the material in the chapter is familiar from previous courses; it is repeated here for purposes of review and completeness. You will probably find that the material of Sections 2-1, 2-2, 2-3, and 2-5 may be covered very quickly. It is likely that the material on polar coordinates, direction on a line, angles between lines, and the normal and polar forms of an equation of a line will be new to most students. The majority of the class time should be spent on these topics. Many examples have been interspersed throughout the text. Though these increase the number of pages in the chapter, hopefully they will help the student to proceed more rapidly and decrease the need for classroom explanation and discussion. Many more exercises have been included than any given class might be expected to do. You will probably find it advisable to break the chapter into two units for testing purposes. For this reason a set of review exercises has been included after Section 2-5.

7-15 If the students are to get anything out of this section, they must understand clearly the treatment of distance in SMSG Geometry. By the Distance Postulate, to every pair of different points there corresponds a unique positive number. It is called the distance between the points because it is the "official" version of the intuitive notion of distance. The Ruler and Ruler Placement Postulates enable us to make any point on a line the origin of a coordinate system, and to make either direction from that point the positive one. However, we can not choose the scale. It is already there in the geometry. Betweenness and congruence are defined in terms of coordinates, and thus coordinate systems are fundamental in the development of the SMSG Geometry.

Nevertheless, intuition tells us that scale doesn't really matter. If two boats are equally long, their lengths expressed in meters are equal just as their lengths expressed in feet are equal. Let a, b, and c be the coordinates of the points A, B, and C on a line, in a certain coordinate system, and $a < b < c$. Then if we change the size of the units (but nothing else) in our coordinate system, and a', b', and c' are the new coordinates of the same points, we should find that $a' < b' < c'$. We have not attempted to prove that we do have this freedom in the text. In order to get started on the task before us, we have offered examples illustrating the ways in which we normally assume this freedom in applying geometry. The examples themselves are trivial in difficulty and were deliberately chosen so; their purpose is to illustrate the many assumptions we make in solving even a simple problem as well as the importance of these assumptions.

9 The techniques of analytic geometry are more saleable if we exploit to the fullest the freedom to choose various coordinate systems. When the occasions arise to mention this freedom, we shall make much of it, usually by invoking a grandiose principle as we do here in the Linear Coordinate System Principle.

In this principle we are actually postulating a theorem we could prove, but the proof is difficult for most students. We have included material in the supplement to Chapter 2, for able students who are well versed in SMSG Geometry and the concept of function, and who are interested in the deductive nature of mathematics.

Note that the symbol "d(R,S)" is defined in terms of a fixed coordinate system. It would be nice if our notation showed this, but that would make it rather complicated. It is advisable to stress this point when the symbol is introduced, so the students will be reminded of it every time they see it later.

10 The definition of a directed segment will probably seem rather unnatural to the students. They will feel that the idea of the segment $\overline{AB}$ considered as running from A to B is quite clear and they will wonder why we give this strange definition. It may help to ask them to try to define the concept in terms which are "official" in our formal system. They will find that any definition of this kind, and no other kind is permissible, seems unnatural.

This is not the first time the students have seen such a definition. They undoubtedly felt they knew what the inside of a triangle was before they studied geometry, and most of them were probably surprised to find out how much trouble it was to give an acceptable definition.

Exercises 2-1

1. There should be some agreement between the numbers obtained by comparing these measurements and those numbers in the text. However, the degree of agreement will depend upon how well the subdivisions of the units are estimated. The constants of proportionality should be consistent.

2. The side is measured to 2 place accuracy and the results are correct to 2 place accuracy. The discrepency between 2.53 and 2.54 is not significant because they are the same to 2 place accuracy.

3. Hopefully, students will be able to anticipate that the proper units are feet; the computed answer (12π ft. $= 37.6992$) seems so idealized to be meaningless.

4. The answer will depend upon the source of the information as to the distance from New York to San Francisco. The answer should be close to 400 miles to the inch.

5. 1 inch represents ≈ 330 miles; the "line" from New York to San Francisco would be approximately 9.2 inches long.

6. The bicyclist travels at the rate of 8 mi/hour. The friend travels at the rate of 32 km/hour or ≈ 20 mi/hour.
   a) $8t - 20(t - 2)$ = distance apart at time t. One hour after the friend begins $(t = 3)$ the distance apart is 4 miles.
   b) When the distances both have traveled are equal, $20(t - 2) = 8t$ and $t = 3\frac{1}{3}$ hours. The distance is (approximately) 27 miles.

7. Rate of bicyclist A is 4 miles/hour.
   Rate of bicyclist B is 5 miles/hour.
   Rate of preposterous bee is 10 miles/hour.
   a)
   $$10t + 5t = 30$$
   $$15t = 30$$
   $$t = 2 \text{ hour}$$
   Distance bee traveled $= 2 \times 10 = 20$ mi.
   b)
   $$4t + 5t = 30$$
   $$9t = 30$$
   $$t = \frac{10}{3} \text{ or } 3\frac{1}{3} \text{ hours}$$
   Total distance bee traveled $= 3\frac{1}{3} \times 10$ or $33\frac{1}{3}$ mi.

16 The statement of the Linear Coordinate System Principle clearly indicates that the measures of distance are proportional, but it is perhaps not so clear that the criteria for order, or betweenness, also carry over in the coordinate systems which we consider. It is not a trivial matter to show that it does. Unfortunately, any numerical example would be hopelessly artificial. An illustration of this idea can be found in physics. The boiling point of alcohol is between the boiling point of water and the freezing point of water. The relationship of betweenness would hold for the corresponding temperatures at these points, whether indicated in the Fahrenheit or the Centigrade scales.

18 The notion of a point of division may be extended to include the endpoints of the segment and points external to the segment, but directed distance should be used in this case in order to assure uniqueness. If in the equation $\frac{\overrightarrow{d}(P,X)}{\overrightarrow{d}(P,Q)} = t$, we define $\overrightarrow{d}(P,X)$ to be the directed distance from P to X and $\overrightarrow{d}(P,Q)$ to be the directed distance from P to Q, we may write

$$\frac{x - p}{q - p} = t .$$

In this case, when $0 < t < 1$, we still obtain internal points of division. When $t = 0$, we obtain the coordinate of P; when $t = 1$, we obtain the coordinate of Q. When $t < 0$, we obtain the coordinates of points in the ray $\overrightarrow{QP}$ which are external to $\overline{PQ}$; when $t > 1$, we obtain points in the ray $\overrightarrow{PQ}$ which are external to $\overline{PQ}$.

18 If your students are like ours, they will comprehend the notion of a weighted average even more clearly when it is applied to test grades which are "weighted" in calculating the final average.

20 There is additional material on linear combinations in the Supplement to Chapter 3 and in SMSG Intermediate Mathematics on pages 374-376 and page 449.

The parametric representation is equivalent to the extension of the notion of point of division given in the note on page 18. If the SMSG Geometry with Coordinates is available, you may wish to look at the material on pages 107-111.

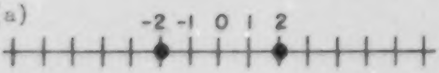
21 The material on the analytic representations of the subsets of a line is more important as an introduction to later work than it is in itself. It provides a review of the notion of the graph of an equation and a reminder

that conditions other than equations also have graphs. If the students are not familiar with the properties of inequalities, it may be necessary to spend a little time on them at this point.

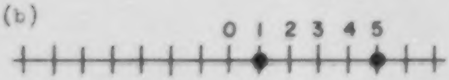
Exercises 2-2

1\.

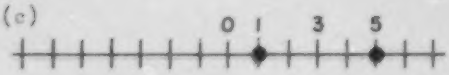
(a)



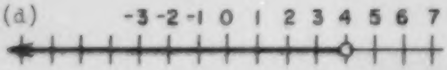
(b)



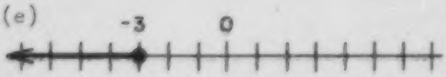
(c)



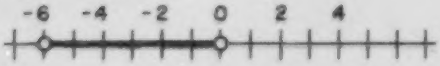
(d)


-3
-2
-1
0
1
2
3
4
5
6
7

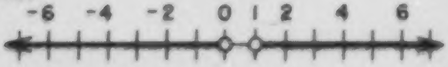
(e)



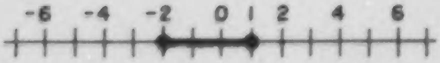
(f)



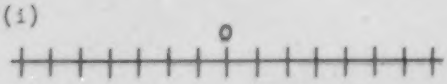
(g)



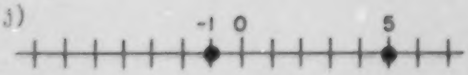
(h)



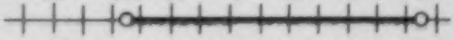
(i)



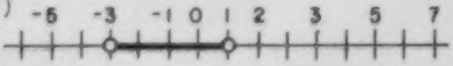
(j)



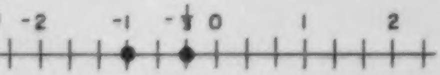
(k)


0.4
0.667
0.8
1.0
1.2
1.4
1.679

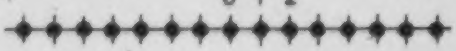
(l)



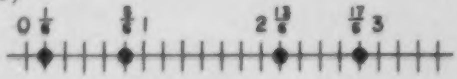
(m)



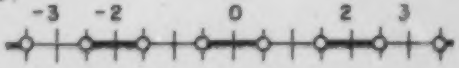
(n)



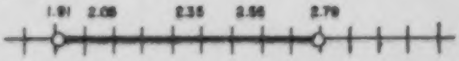
(o)


0
1/6
5/6
1
2
13/6
17/6
3

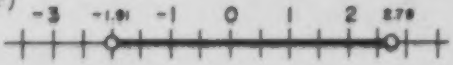
(p)



(q)


1.91
2.08
2.35
2.56
2.79

(r)


-3
-1.91
-1
0
1
2
2.79

2. (a) $3 \leq x \leq 4$
   Alternative: $(x-3)(x-4) \leq 0$

   (b) $-2 \leq x < 2$

   (c) if $b > a$:
   $x \geq a + 2(b - a) = 2b - a$

   (d) $x \leq x_2 + \frac{1}{2}(x_2 - x_1)$

   (e) $(x+1)(x)(x-1)(x-3)\frac{x}{x} \leq 0$
   Alternative: $-1 \leq x < 0$
   or $1 \leq x \leq 3$

   (f) $(x+2)(x+1)(x-1)(x-3)\frac{(x-1)}{(x-1)} \leq 0$

   (g) $|x - \frac{2}{5}| \leq \frac{1}{5}$
   Alternative: $\frac{1}{5} \leq x \leq \frac{3}{5}$

   (h) $x < 3$

   (i) $\sin \pi x \neq 0$

   (j) $\sin \theta \geq 0$

3. (a) $3a$, $-3a$

   (b) All values of x such that $0 \leq x \leq 1$

4. (a) $m = \frac{15}{2}$ $\quad a = 6$ $\quad b = 9$

   (b) $m = \frac{11}{2}$ $\quad a = 3$ $\quad b = 8$

   (c) $m = r$ $\quad a = r + \frac{1}{3}s$ $\quad b = r - \frac{1}{3}s$

   (d) $m = (r + t) + 1$ $\quad a = (r + t)$ $\quad b = (r + t) + 2$

   (e) $m = r + \frac{3}{2}t$ $\quad a = \frac{4}{3}r + t$ $\quad b = \frac{2}{3}r + 2t$

   (f) $m = \frac{5}{2}r + \frac{1}{2}s$ $\quad a = \frac{7}{3}r + \frac{4}{3}s$ $\quad b = \frac{8}{3}r - \frac{1}{3}s$

   (g) $m = \frac{1}{2}(r^2 - r + s^2 - s)$ $\quad a = \frac{2}{3}(r^2 - r) + \frac{1}{3}(s^2 - s)$ $\quad b = \frac{1}{3}(r^2 - r) + \frac{2}{3}(s^2 - s)$

   (h) $m = \frac{1}{2}(r + s)$ $\quad a = \frac{2}{3}r + \frac{1}{3}s$ $\quad b = \frac{1}{3}r + \frac{2}{3}s$

5. (a) $X = Q$

   (b) $X = P$

   (c) X is between P and Q

   (d) Q is between P and X

   (e) P is between X and Q

   (f) Q is between P and X

6. (a) $t = -1$

$t = \frac{1}{3}$

(b) $t = \frac{2}{3}$

$t = 2$

(c) $t = 3$

$t = -1$

(d) $t = -1$

$t = 1$

7. (a) $\frac{d(A,B)}{d(B,C)} = \frac{1 - 1\frac{1}{2}}{1\frac{1}{2} - 2\frac{1}{2}} = \frac{-\frac{1}{2}}{-1} = \frac{1}{2}$

(b) $\frac{d(B,C)}{d(C,D)} = \frac{1\frac{1}{2} - 2\frac{1}{2}}{2\frac{1}{2} - 4\frac{1}{2}} = \frac{-1}{-2} = \frac{1}{2}$

(c) $\frac{d(C,D)}{d(D,E)} = \frac{2\frac{1}{2} - 4\frac{1}{2}}{4\frac{1}{2} - 9} = \frac{-2}{-4\frac{1}{2}} = \frac{4}{9}$

8. (a) $b = \frac{2}{3}a + \frac{1}{3}c$

(b) $c = \frac{2}{3}b + \frac{1}{3}d$

(c) $d = \frac{9}{13}c + \frac{4}{13}e$

9. (a) $T_1 = 1\frac{1}{2}$ $T_2 = 2$

(b) $T_1 = 2\frac{1}{2}$ $T_2 = 3\frac{1}{2}$

(c) $T_1 = \frac{14}{3}$ $T_2 = \frac{41}{6}$

10. $P = \frac{3}{4}$ or $\frac{9}{4}$

$Q = 1$ or 4

$R = 6$ or 12

26 The teacher will have to use his own judgment as to how much time should be spent on coordinate systems in the plane not of the type we define. For example, if we consider two mutually perpendicular lines and on each of them a perfectly arbitrary linear coordinate system, then by the method described in the text there is established a one-to-one correspondence between the points in the plane and the ordered pairs of real numbers. However, many things become more complicated. The distance between two points, for example, is no longer given by the usual formula. Probably no more than a few minutes should be spent on this in class, after which Challenge Exercise 4 on page 54 can be assigned. (See Supplement C for more on this subject.)

27 We may, of course, extend the notion of point of division as we did on page 18.

29 If the SMSG Geometry with Coordinates is available, you may want to look at pages 543-550, where there is an alternative development of the parametric representation of the points on a line.

Exercises 2-3

1. (a) $M = (3, 4\frac{1}{2})$

   $A = (2,3)$

   $B = (4,6)$

   (b) $M = (5, 7\frac{1}{2})$

   $A = (4,6)$

   $B = (6,9)$

   (c) $M = (5\frac{1}{2}, 2\frac{1}{2})$

   $A = (5\frac{1}{3}, 5\frac{2}{3})$

   $B = (5\frac{2}{3}, -\frac{2}{3})$

   (d) $M = (-2\frac{1}{2}, 3\frac{1}{2})$

   $A = (-\frac{1}{3}, 1\frac{1}{3})$

   $B = (-\frac{14}{3}, \frac{17}{3})$

(e) $M = (0,0)$

$A = (-2,-1)$

$B = (2,1)$

(f) $M = (-4\frac{1}{2}, -4\frac{1}{2})$

$A = (-4,-5)$

$B = (-5,-4)$

(g) $M = \frac{p_1 + q_1}{2}, \frac{p_2 + q_2}{2}$

$A = \frac{2p_1 + q_1}{3}, \frac{2p_2 + q_2}{3}$

$B = \frac{p_1 + 2q_1}{3}, \frac{p_2 + 2q_2}{3}$

(h) $M = (\frac{3s}{2}, \frac{3t}{2})$

$A = (\frac{5s}{3}, \frac{8t}{3})$

$B = (\frac{4s}{3}, \frac{t}{3})$

(i) $M = (\frac{3r}{2} + \frac{s}{2}, -2r - \frac{s}{2})$

$A = (\frac{7r}{3} + s, -\frac{7r}{3})$

$B = (\frac{2r}{3}, -\frac{5r}{3} - s)$

2. (a) $x = 2a + 6b$

$y = 3a + b$

(b) $x = -4a + 2b$

$y = 5a - 7b$

(c) $x = -3a - 6b$

$y = -6a + 4b$

3. (a) $x = 2 + 4t$

$y = 3 - 2t$

(b) $x = -4 + 6t$

$y = 5 - 12t$

(c) $x = -3 - 3t$

$y = -6 + 10t$

4. If, in equation (2) , $x_0 = x_1$ or $y_0 = y_1$

$$x = \frac{dx_0 + cx_0}{c + d}$$

or

$$y = \frac{dy_0 + cy_0}{c + d}$$

Simplifying, $x = x_0$

or $y = y_0$

These are conditions describing points on lines parallel to the y-axis or x-axis respectively.

5. (a) Substituting into equation (1) we see that

$$\frac{7 - (-3)}{22 - (-3)} = \frac{0 - (-6)}{9 - (-6)}$$

$$\frac{10}{25} = \frac{6}{15}$$

$$\frac{2}{5} = \frac{2}{5}$$ $\therefore$ Points A , B , C are collinear

Check:

$$d(A,B) = \sqrt{(7 - (-3))^2 + (0 - (-6))^2}$$
$$= \sqrt{136} = 2\sqrt{34}$$

$$d(B,C) = \sqrt{((-3) - 22)^2 + ((-6) - 9)^2}$$
$$= \sqrt{850} = 5\sqrt{34}$$

$$d(A,C) = \sqrt{(7 - 22)^2 + (0 - 9)^2}$$
$$= \sqrt{306} = 3\sqrt{34}$$

$$d(A,B) + d(A,C) = 2\sqrt{34} + 3\sqrt{34} = d(B,C)$$

$\therefore$ A, B, C must be collinear

(b)

$$\frac{-1 - 3}{-5 - 3} \overset{?}{=} \frac{4 - (-14)}{-6 - (-14)}$$

$$\frac{-4}{-8} \neq \frac{18}{8}$$ not collinear

Check:

$$d(A,B) = \sqrt{((-1) - 3)^2 + (4 -(-14))^2}$$
$$= \sqrt{340} = 2\sqrt{85}$$

$$d(B,C) = \sqrt{(3 - (-5))^2 + ((-14) - (-6))^2}$$
$$= \sqrt{128} = 8\sqrt{2}$$

$$d(A,C) = \sqrt{((-1) - (-5))^2 + (4 - (-6))^2}$$
$$= \sqrt{592} = 4\sqrt{37}$$

$$d(A,B) + d(B,C) \neq d(A,C)$$

This verifies that the points are not collinear.

6. Given that: A (1,-1),
B (4,7), and
P (h,-3)

$$\frac{1-4}{h-4} = \frac{-1-7}{-3-7}$$

$$\frac{-3}{h-4} = \frac{-8}{-10}$$

$$-8h + 32 = 30$$

$$-8h = -2$$

$$h = \frac{1}{4}$$

30-38 Polar coordinates are a new topic for most students and care must be taken in their presentation. The primary difficulty is the multiplicity of the polar representations of a given point.

31 Other examples of the physical application of polar coordinates occur in air and sea navigation. The path of a racing sail boat beating up to a mark may appeal to some students. The paths across newly planted lawns on corner lots bear this out, too.

31 In the definition of the polar angle it may be necessary to stress that the terminal ray of the angle need not contain the point. This is a recurrent pitfall in verbal descriptions. The angle POM is not the only polar angle of the point P .

32 The fact that (r,θ) and $(-r,\theta + \pi)$ both represent the same point is worthy of emphasis. A student of the calculus must exercise particular care in the use of polar coordinates. If a curve is symmetric with respect to the origin, it is all too easy to sum up the area bounded by the curve on one side of the origin--and at the same time subtract away an equal area on the other. A judicious use of symmetry and boundaries is essential in such cases.

35 Once again we want to stress the freedom to choose our analytic framework in any way which will make algebraic manipulation as painless as possible. In general, if P and Q are any two distinct points in any plane and if (p_1,p_2) and (q_1,q_2) are any two distinct ordered pairs of real numbers, there exists a rectangular coordinate system in that plane in which $P = (p_1,p_2)$ and $Q = (q_1,q_2)$. Furthermore, if we let (r_1,θ_1) and (r_2,θ_2) be any two distinct ordered pairs of real numbers, there exists a polar coordinate system in the plane in which $P = (r,\theta_1)$ and $Q = (r_2,\theta_2)$. (Note that the change from (p_1,p_2) and (q_1,q_2) to (r_1,θ_1) and (r_2,θ_2) was

unnecessary; any two distinct ordered pairs of real numbers may be coordinates of P and Q in coordinate systems of each type. If at least one of the points is not on an axis, the coordinate system is unique.)

35 A moment's thought should convince you that the usual equations relating polar and rectangular coordinates are completely dependent upon a particular orientation of both coordinate systems in the same plane. If either coordinate system should be introduced differently into the plane, we would have to develop new equations of transformation.

36 The ordered pairs (r,θ) satisfying equations (2) describe two distinct points, but once the student has developed some facility with polar coordinates, it will be easy to choose the appropriate ones. If the students are familiar with the inverse trigonometric relations, they may prefer some equivalent of the follwoing definition,

$$P = \{(r,\theta) : \text{ where } r = \pm\sqrt{x^2 + y^2} \neq 0 \text{ , } \theta = \cos^{-1}\frac{x}{r}$$
$$= \sin^{-1}\frac{y}{r} \text{ ;}$$

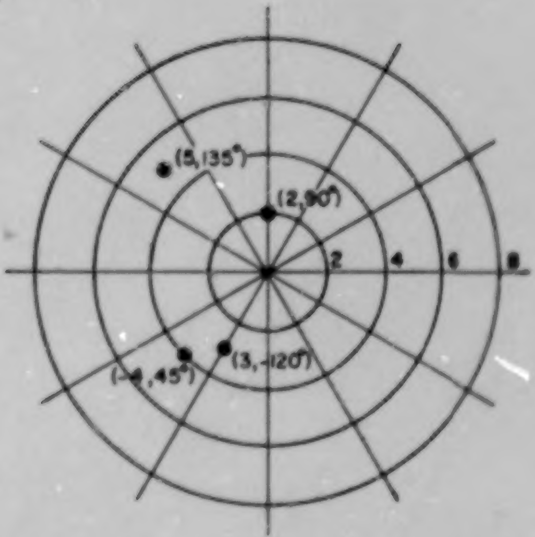
where $x^2 + y^2 = 0$, $r = 0$ and θ is any real number.} Hopefully, a student will ask what to do when $x = 0$, since one of the equations of transformation is not defined. Some other student should be able to point out that in this case $\theta = \frac{\pi}{2} + n\pi$, where n is any integer.

37 Example 5 is worth some attention, for the application of the Law of Cosines as a distance formula in polar coordinates is often convenient. Again there is a loophole, for it may not be apparent that the Law of Cosines still applies if $\theta_1 = \theta_2 + n\pi$, where n is any integer. In Section 2-7 we shall have occasion to point out that the relationship described still holds even when the "vertices of the triangle" are collinear.

38-40 There is a wealth of practice exercises here. Exercise 5 would require seventy different answers if all parts were done; Exercise 10 has over thirty answers. You will probably want to pick and choose within this set of exercises, but there is plenty of extra drill available for students who need it.

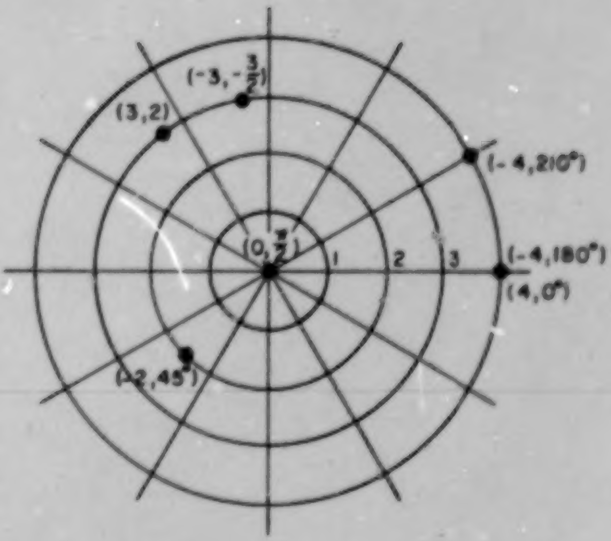
Exercises 2-4

1.

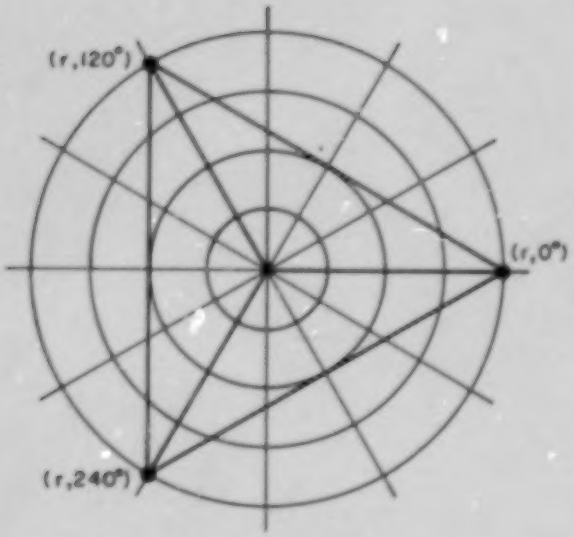

(5,135°)
(2,90°)
2
4
6
8
(-4,45°)
(3,-120°)

(5,135°)(-5,315°)
(5,495°)
(-5,-45°)
(2,90°)(-2,270°)
(2,450°)
(-2,-90°)
(-4,45°)(4,-135°)
(4,225°)
(-4,405°)
(3,-120°) (-3,60°)
(3,240°)
(3,600°)

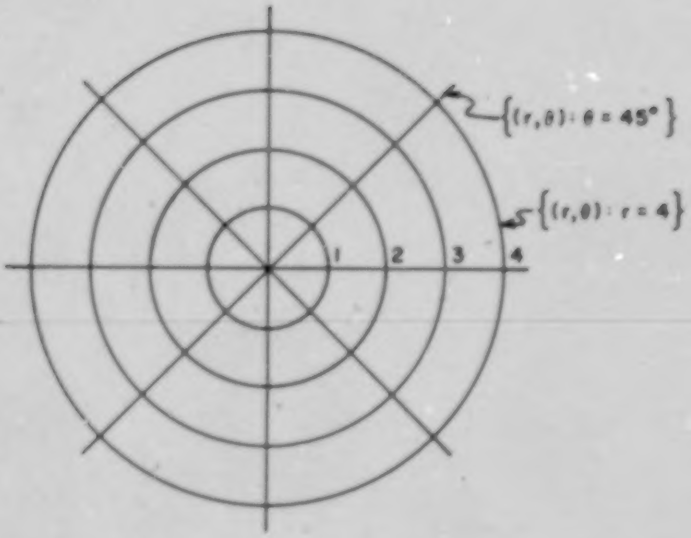
2.


$(-3, -\frac{3\pi}{2})$
(3,2)
(-4,210°)
$(0, \frac{\pi}{2})$
1
2
3
(-4,180°)
(4,0°)
(-2,45°)

3.


(r, 120°)
(r, 0°)
(r, 240°)

4.


{(r, θ): θ = 45°}
{(r, θ): r = 4}
1
2
3
4

5.	A$(2,270^\circ)$	$(2,-90^\circ)$	$(-2,90^\circ)$	$(2,\frac{3\pi}{2})$	$(-2,90^\circ)$
	B$(3,300^\circ)$	$(3,-60^\circ)$	$(-3,120^\circ)$	$(3,\frac{5\pi}{3})$	$(-3,120^\circ)$
	C$(4,330^\circ)$	$(4,-30^\circ)$	$(-4,150^\circ)$	$(4,\frac{11\pi}{6})$	$(-4,150^\circ)$
	D$(5,0^\circ)$	$(5,-0^\circ)$	$(-5,180^\circ)$	$(5,0)$	$(5,0^\circ)$
	E$(6,30^\circ)$	$(6,-330^\circ)$	$(-6,210^\circ)$	$(6,\frac{\pi}{6})$	$(6,30^\circ)$
	F$(7,60^\circ)$	$(7,-300^\circ)$	$(-7,240^\circ)$	$(7,\frac{\pi}{3})$	$(7,60^\circ)$
	G$(8,90^\circ)$	$(8,-270^\circ)$	$(-8,270^\circ)$	$(8,\frac{\pi}{2})$	$(8,90^\circ)$
	H$(9,120^\circ)$	$(9,-240^\circ)$	$(-9,300^\circ)$	$(9,\frac{2\pi}{3})$	$(9,120^\circ)$
	I$(10,150^\circ)$	$(10,-210^\circ)$	$(-10,330^\circ)$	$(10,\frac{5\pi}{6})$	$(10,150^\circ)$
	J$(\frac{3}{2},180^\circ)$	$(\frac{3}{2},-180^\circ)$	$(-\frac{3}{2},0^\circ)$	$(\frac{3}{2},\pi)$	$(-\frac{3}{2},0^\circ)$
	K$(\frac{5}{2},210^\circ)$	$(\frac{5}{2},-150^\circ)$	$(-\frac{5}{2},30^\circ)$	$(\frac{5}{2},\frac{7\pi}{6})$	$(-\frac{5}{2},30^\circ)$
	L$(\frac{7}{2},240^\circ)$	$(\frac{7}{2},-120^\circ)$	$(-\frac{7}{2},60^\circ)$	$(\frac{7}{2},\frac{4\pi}{3})$	$(-\frac{7}{2},60^\circ)$
	M$(5,285^\circ)$	$(5,-75^\circ)$	$(-5,105^\circ)$	$(5,\frac{19\pi}{12})$	$(-5,105^\circ)$
	N$(6,315^\circ)$	$(6,-45^\circ)$	$(-6,135^\circ)$	$(6,\frac{7\pi}{4})$	$(-6,135^\circ)$

6. (a) $(0,0)$
   (b) $(1,-1)$
   (c) $(\frac{5}{2},\frac{5}{2}\sqrt{3})$
   (d) $(4,0)$
   (e) $(-1,0)$
   (f) $(0,\sqrt{2})$
   (g) $(-1,-\sqrt{3})$
   (h) $(\sqrt{2},-\sqrt{2})$

7. (a) $(\sqrt{2},45^\circ)$
   (b) $(2\sqrt{2},315^\circ)$
   (c) $(p,0^\circ)$
   (d) $(q,\frac{\pi}{2})$
   (e) $(2,150^\circ)$
   (f) $(2,240^\circ)$
   (g) $(\sqrt{29},22^\circ)$
   (h) $(\sqrt{17},166^\circ)$

8. (a) $d(A,B)$ when $A = (2,150^\circ)$ and $B = (4,210^\circ)$

$$= \sqrt{(2)^2 + (4)^2 - 2(2)(4)\cos(210^\circ - 150^\circ)} = 2\sqrt{3}$$

Using rectangular coordinates

$A = (2,150^\circ)$ in rectangular coordinates $(-\sqrt{3},1)$

$B = (4,210^\circ)$ in rectangular coordinates $(-2\sqrt{3},-2)$

$$d(A,B) = \sqrt{(-\sqrt{3} - (-2\sqrt{3}))^2 + (1 - (-2))^2}$$

$$= \sqrt{(\sqrt{3})^2 + (3)^2} = 2\sqrt{3}$$

(b) Using rectangular coordinates:

$A = (5, \frac{5}{4}\pi)$ in rectangular coordinates $(-\frac{5}{2}\sqrt{2}, -\frac{5}{2}\sqrt{2})$

$B = (12, \frac{7}{4}\pi)$ in rectangular coordinates $(6\sqrt{2}, -6\sqrt{2})$

9. (a) $d(A,B) = \sqrt{34}$

(b) $A = (2,37^\circ)$, $B = (3,100^\circ)$

$d(A,B) = \sqrt{4 + 9 - 2(2)(3)\cos(100 - 37)}$

$d(A,B) = \sqrt{4 + 9 - 12(.454)}$

$d(A,B) = \sqrt{4 + 9 - 5.45} = \sqrt{7.55} = 2.75$

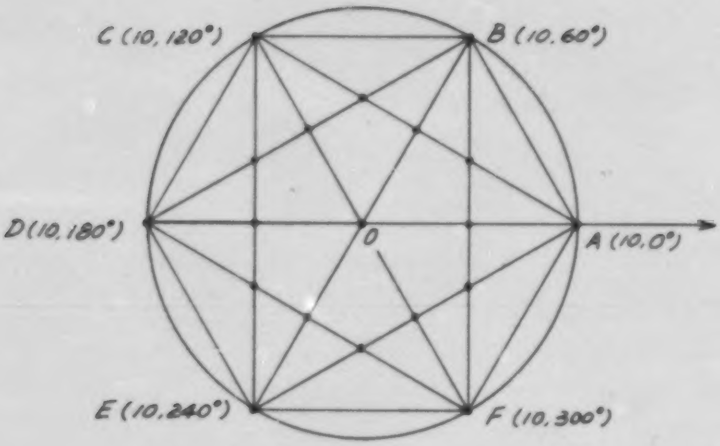
(c) $d(A,B) = \sqrt{52}$

(d) $d(A,B) = \sqrt{7}$

(e) $d(A,B) = 7$

(f) $d(A,B) = 5\sqrt{5}$

10.



	$\overline{DF}$	$\overline{CF}$	$\overline{CE}$	$\overline{BF}$	$\overline{BE}$	$\overline{BD}$	$\overline{AD}$	$\overline{AC}$
$\overline{AC}$	--	C	C	$(\frac{10}{3}\sqrt{3}, 30^\circ)$	$(5, 60^\circ)$	$(\frac{10}{3}\sqrt{3}, 90^\circ)$	A	
$\overline{AD}$	D	O	$(5, 180^\circ)$	$(5, 0^\circ)$	O	D		
$\overline{AE}$	$(\frac{10}{3}\sqrt{3}, 270^\circ)$	$(5, 300^\circ)$	E	$(\frac{10}{3}\sqrt{3}, 330^\circ)$	E	--		
$\overline{BD}$	D	$(5, 120^\circ)$	$(\frac{10}{3}\sqrt{3}, 150^\circ)$	B	B			
$\overline{BE}$	$(5, 240^\circ)$	O	E	B				
$\overline{BF}$	F	F	--					
$\overline{CE}$	$(\frac{10}{3}\sqrt{3}, 210^\circ)$	C						
$\overline{CF}$	F							
$\overline{DF}$								

This chart shows the points of intersection of the diagonals of a hexagon inscribed in circle with radius 10 one vertex at $(10, 0^\circ)$.

The twelve interior points of intersection different from O are

$(\frac{10}{3}\sqrt{3}, 30^\circ)$	$(\frac{10}{3}\sqrt{3}, 90^\circ)$	$(\frac{10}{3}\sqrt{3}, 150^\circ)$
$(\frac{10}{3}\sqrt{3}, 210^\circ)$	$(\frac{10}{3}\sqrt{3}, 270^\circ)$	$(\frac{10}{3}\sqrt{3}, 330^\circ)$
$(5, 0^\circ)$	$(5, 60^\circ)$	$(5, 120^\circ)$
$(5, 180^\circ)$	$(5, 240^\circ)$	$(5, 300^\circ)$

11. (a) $((-1)^k r_0, (\theta_0 + 180k)^\circ)$
    (b) $((-1)^k r_0, \theta_0 + \pi k)$

41-49 Students should find little if any new material in this section. It is included for review and completeness.

41 The geometric form is useful in developing equations for a line, since it is closely allied both to the geometric picture and, since the denominators are direction numbers for the line, to the parametric representation for the line. It corresponds to the symmetric equations for a line in 3-space.

43 Inclination is defined geometrically, since our point of view is geometric. This definition may also prepare the student for the definition of direction angles in the following section.

44 Note that inclination is defined even when slope is not.

49 Since the general form of an equation of a line does not reveal immediately the geometric characteristics of the line, it is worthwhile to develop facility in interpreting the geometric properties from the coefficients.

Exercises 2-5

1. $y + 3 = 2(x - 2)$ $2x - y - 7 = 0$ $p = 7$ $q = 3$
2. $y - 5 = -\frac{2}{3}(x + 3)$ $p = -6$ $q = -\frac{1}{3}$
3. $y = 3x + b$ $p = \frac{7 - b}{3}$ $q = 15 + b$
4. $y - 5 = \frac{2}{3}(x - 4)$ The two lines are parallel.
5. $y = k(x - a)$ y-intercept at $(0, -ka)$
6. $ax + by = 0$ a, b real numbers.
   $5x + 3y = 0$ contains $(-3, 5)$
7. Slope of $\overleftrightarrow{OA}$ is $\frac{5}{3}$ slope of $\overleftrightarrow{OB}$ is $-\frac{3}{5}$

   Two lines are perpendicular if and only if
   (a) the product of their slopes is -1 or
   (b) one has no slope and the other zero slope.
8. $\frac{x + 8}{4} = \frac{y - 8}{-3}$

9. (1) $\frac{x + 4}{6} = \frac{y - 8}{-5}$

(2) $5x + 6y - 28 = 0$

(3) $y - 8 = -\frac{5}{6}(x + 4)$

(4) $y = -\frac{5}{6}x + \frac{14}{3}$

(5) $\frac{y}{\frac{14}{3}} + \frac{x}{\frac{28}{5}} = 1$

(6) $y - 8 = \frac{3 - 8}{2 + 4}(x + 4)$

(7) $x + 4 = \frac{2 + 4}{3 - 8}(y - 8)$

Slope: $-\frac{5}{6}$ x-intercept: $\frac{28}{5}$ y-intercept: $\frac{14}{3}$

$$y = -\frac{a}{b}x - \frac{c}{b}$$

10. (a) If $b = 0$, $ac \neq 0$, line is vertical, through $(-\frac{c}{a}, 0)$

(b) If $a = 0$, $bc \neq 0$, line is horizontal, through $(0, -\frac{c}{b})$

(c) If $c = 0$, $ab \neq 0$, line has slope $-\frac{a}{b}$, through $(0,0)$

11. (a) $y = -\frac{7}{3}x + 5$

(b) $y = x - 5$

(c) $y = -\frac{2}{7}x + \frac{17}{7}$

(d) $y = -x - 2$

(e) $y = \frac{\sqrt{3}}{3}x + \frac{\sqrt{3} - 9}{3}$

12. (a) $\frac{x - 3}{1 - 3} = \frac{y - 2}{-2 - 2}$

(b) The midpoint of $\overline{BC}$ is $(\frac{3}{2}, 3)$

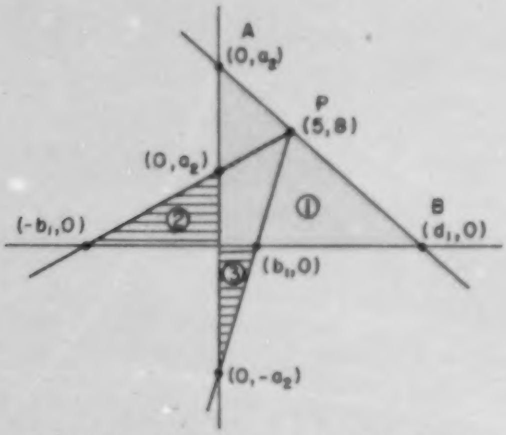
Median from A can be represented by

$$\frac{x - 1}{\frac{3}{2} - 1} = \frac{y - (-2)}{3 - (-2)}\text{, or } 10x - y - 12 = 0 .$$

(c) The midpoint of $\overline{AC}$ is $(\frac{1}{2}, 1)$. And from (b) midpoint of $\overline{BC}$ is $(\frac{3}{2}, 3)$. Line joining these two points is represented by

$$\frac{x - \frac{1}{2}}{\frac{3}{2} - \frac{1}{2}} = \frac{y - 1}{3 - 1}\text{, or } 2x - y = 0 .$$

13. Given the conditions of the problem, it appears that there are three possible solutions. (sketch below)



Triangle ①: This triangle is not satisfactory, since its area must be greater than 40 ; that is, its area includes that of the rectangle with O and P as opposite vertices, and adjacent sides on the axes.

Triangle ②: The area of the triangle is $\frac{1}{2}a_2b_1$. The slope of $\overline{BP}$ = slope of $\overline{AB}$ and

$$\frac{8}{5 + b_1} = \frac{a_2}{b_1}$$

Solving for a_2 ,

$$a_2 = \frac{8b_1}{5 + b_1}$$

Substituting into $\frac{1}{2}a_2b_1$, we find that the positive root is $5 (b_1 = 5)$.

Using $a_2 = \frac{8b_1}{5 + b_1}$, we find $a_2 = 4$.

The equation of the line through (0,4) , (-5,0) , and (5,8) using the symmetric form is

$$\frac{x + 5}{0 + 5} = \frac{y - 0}{4 - 0}, \text{ or } 4x - 5y + 20 = 0 .$$

Triangle ③: Area of triangle 3 is $\frac{1}{2}a_2 b_1$.

Slope of $\overline{PB}$ = slope of $\overline{AB}$

$$\frac{8}{5 - b_1} = \frac{a^2}{b_1}$$

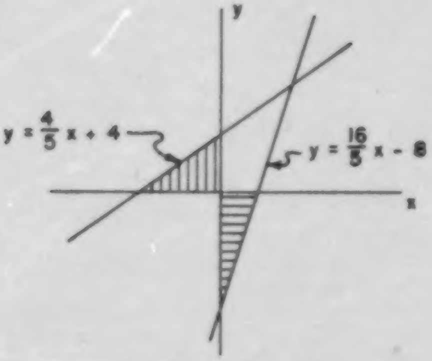
Solving for a_2 , we see that

$$a_2 = \frac{8b_1}{5 - b_1}$$

Substituting in area formula, $b_1 = \frac{20}{8}$ and $a_2 = 8$.

The equation of the line through $(0,-8)$, $(\frac{20}{8}, 0)$ and $(5,8)$ in symmetric form is

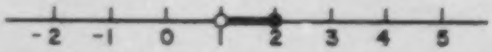
$$\frac{x - 0}{5 - 0} \quad \frac{y - (-8)}{8 - (-8)} \text{, or } 16x - 5y - 40 = 0 \text{ .}$$



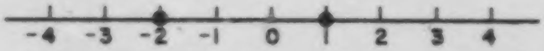
Since this is such a long chapter, you may want to test the students at this point. With this in mind we have included a copious set of review and challenge exercises from which selections may be made.

Review Exercises - Section 2-1 through Section 2-5

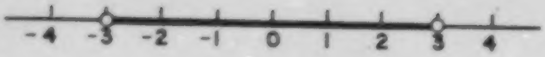
1. $\{x: 1 < x \leq 2\}$



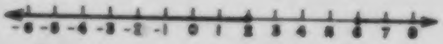
2. $\{x: (x - 1)(x + 2) = 0\}$



3. $\{x: |x| < 3\}$

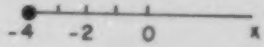


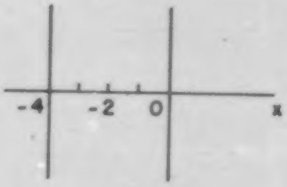
4. $\{x: |x - 4| \geq 2\}$


-6
-5
-4
-3
-2
-1
0
1
2
3
4
5
6
7
8

5. One-space: A point four units to the left of the origin.

Two-space: A line parallel to the y-axis four units to the left of it.

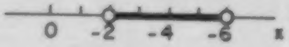


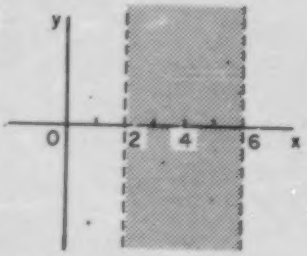


6. The empty set.

7. One-space: A segment of the x-axis between, but not including the points $x = 2$ and $x = 6$.

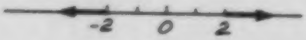
2-space: A portion of the xy-plane between but excluding lines $x = 2$ and $x = 6$.

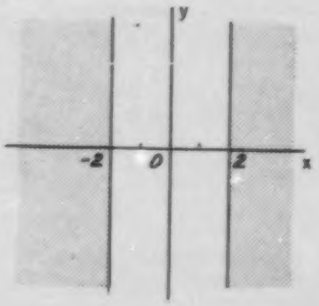




8. One-space: The portion of the x-axis to the right of 2 including $x = 2$ and to the left of and including $x = -2$.

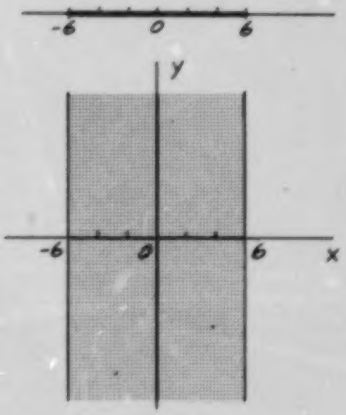
2-space: The portion of the plane to the right of and including the line $x = 2$ and the portion of the plane to the left of and including the line $x = -2$.





9. One-space: A segment of the x-axis between and including the points $x = 6$ and $x = -6$.

2-space: The portion of the plane between and including the lines $x = 6$ and $x = -6$.



10. Let m represent the midpoints and t_1, t_2 represent the trisection points.

(a) $m = \frac{1}{2}$

$t_1 = 0$ and $t_2 = 1$

(b) $m = -2$

$t_1 = -3$ and $t_2 = -1$

(c) $m = \frac{1}{2}$

$t_1 = -\frac{1}{3}$ and $t_2 = 1\frac{1}{3}$

11. (a) $(2, \frac{\pi}{3})$

(b) $(2, \frac{3\pi}{4})$

(c) $(5, -53^\circ)$, approximately.

(d) $(\sqrt{13}, 236^\circ)$, approximately

(e) $(1, 0)$

(f) $(1, \frac{\pi}{2})$

12. (a) $(2\sqrt{2}, 2\sqrt{2})$

(b) $(-\frac{3}{2}, \frac{3\sqrt{3}}{2})$

(c) $(\sqrt{2}, \sqrt{2})$

(d) $(3\sqrt{2}, 3\sqrt{2})$

(e) $(\frac{-5\sqrt{2}}{2}, \frac{-5\sqrt{2}}{2})$

(f) $(\frac{-3\sqrt{3}}{2}, \frac{3}{2})$

13. $3x + 4y = 14$

14. $8x - 11y + 46 = 0$

15. $5x - 2y + 10 = 0$

16. $y = \sqrt{3}x + 5 - 4\sqrt{3}$

17. $y = 6$

18. $x = 4$

19. The equation for $\overleftrightarrow{AB}$ is $y = -\sqrt{3}\ x + 6\sqrt{3}$

The equation for $\overleftrightarrow{BC}$ is $y = 3\sqrt{3}$

The equation for $\overleftrightarrow{CD}$ is $y = \sqrt{3}\ x + 6\sqrt{3}$

The equation for $\overleftrightarrow{DE}$ is $y = -\sqrt{3}\ x - 6\sqrt{3}$

The equation for $\overleftrightarrow{EF}$ is $y = -3\sqrt{3}$

The equation for $\overleftrightarrow{FA}$ is $y = \sqrt{3}\ x - 6\sqrt{3}$

20. The equation for $\overleftrightarrow{AB}$ is $\sqrt{3}\ x + y - 6\sqrt{3} = 0$

The equation for $\overleftrightarrow{BC}$ is $y - 3\sqrt{3} = 0$

The equation for $\overleftrightarrow{CD}$ is $\sqrt{3}\ x - y + 6\sqrt{3} = 0$

The equation for $\overleftrightarrow{DE}$ is $\sqrt{3}\ x + y + 6\sqrt{3} = 0$

The equation for $\overleftrightarrow{EF}$ is $y + 3\sqrt{3}\ y = 0$

The equation for $\overleftrightarrow{FA}$ is $\sqrt{3}\ x - y - 6\sqrt{3} = 0$

21. The equation for $\overleftrightarrow{AB}$ is $\dfrac{x - 6}{-3} = \dfrac{y}{3\sqrt{3}}$

The equation for $\overleftrightarrow{BC}$ is not defined

The equation for $\overleftrightarrow{CD}$ is $\dfrac{x + 6}{3} = \dfrac{y}{3\sqrt{3}}$

The equation for $\overleftrightarrow{DE}$ is $\dfrac{x + 3}{-3} = \dfrac{y + 3\sqrt{3}}{3\sqrt{3}}$

The equation for $\overleftrightarrow{EF}$ is not defined

The equation for $\overleftrightarrow{FA}$ is $\dfrac{x - 3}{3} = \dfrac{y + 3\sqrt{3}}{3\sqrt{3}}$

22. $\dfrac{-\sqrt{3}}{3}$ is the slope of $\overleftrightarrow{AC}$.

$\dfrac{\sqrt{3}}{3}$ is the slope of $\overleftrightarrow{BD}$.

$\dfrac{\sqrt{3}}{3}$ is the slope of $\overleftrightarrow{AE}$.

$\dfrac{-\sqrt{3}}{3}$ is the slope of $\overleftrightarrow{DF}$.

23. Let t_1 and t_2 represent the trisection points.

For $\overline{AB}$, $t_1 = (5,\sqrt{3})$ and $t_2 = (4,2\sqrt{3})$.

For $\overline{BC}$, $t_1 = (1,3\sqrt{3})$ and $t_2 = -1,3\sqrt{3})$.

For $\overline{CD}$, $t_1 = (-4,2\sqrt{3})$ and $t_2 = (-5,\sqrt{3})$.

For $\overline{DE}$, $t_1 = (-5,-\sqrt{3})$ and $t_2 = (-4,-2\sqrt{3})$.

For $\overline{EF}$, $t_1 = (-1,-3\sqrt{3})$ and $t_2 = (1,-3\sqrt{3})$.

For $\overline{FA}$, $t_1 = (4,-2\sqrt{3})$ and $t_2 = (5,-\sqrt{3})$.

24. (a) $P = (4,2\sqrt{3})$ or $(8,-2\sqrt{3})$

(b) $Q = (\frac{3}{7}, 3\sqrt{3})$ or $(21,3\sqrt{3})$

(c) $R = (-\frac{13}{3}, \frac{5\sqrt{3}}{3})$ or $(9,15\sqrt{3})$

25. The inclination of $\overleftrightarrow{AB} = 120^\circ$

The inclination of $\overleftrightarrow{AC} = 150^\circ$

The inclination of $\overleftrightarrow{AE} = 30^\circ$

The inclination of $\overleftrightarrow{AF} = 60^\circ$

26.

Symmetric form.	
displays direction pair	does not exist for lines parallel to either axis
General form.	
always exists	conceals intercepts
displays direction pair	
ease in computing intersections	
ease in telling if L contains (0, 0)	
Point-slope form.	
displays slope	does not always exist
ease in testing if P is on L	
Slope-intercept form.	
displays slope and intercept	does not always exist
Intercept form.	
displays intercepts	does not always exist
displays a direction pair	
Two-point form.	
usual way of finding line through two points	must be used in different form if $\overleftrightarrow{P_1P_2}$ is vertical
determines slope	

(a) general form

(b) intercept form

(c) general form

(d) slope-intercept form

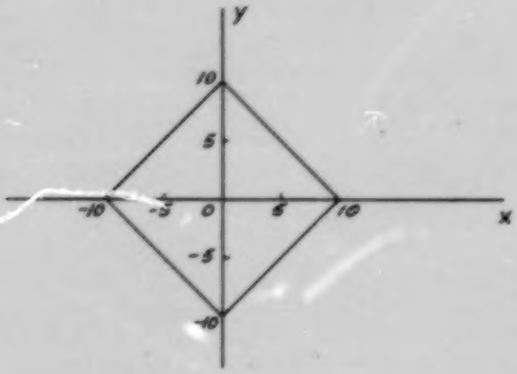
(e) slope-intercept

(f) symmetric

(g) symmetric

(h) symmetric

27. A square as shown in the figure



28. It is interesting to have students note what happens as the constant term shrinks to zero. At this instant the square shrinks to a point. The teacher might ask what happens when the constant is negative.

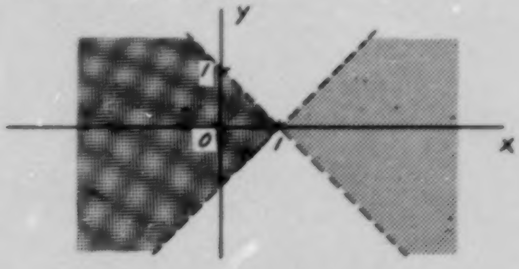
29. The half-plane above and excluding the line $x - y = 1$.

30. The half-plane above and including the line $x - y = 1$.

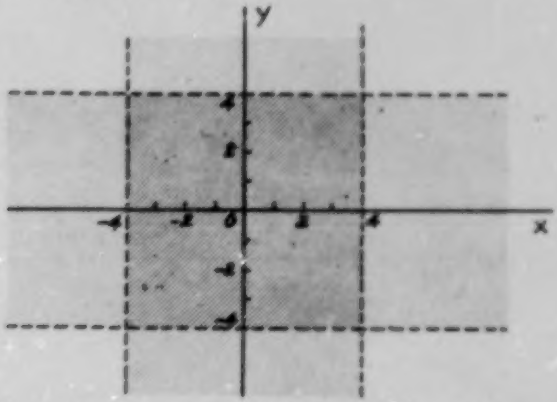
31. The "triangular" portion of the plane below and excluding the lines $x - y = 1$ and $x + y = 1$.

Graph for Exercise 17.

Cross hatch shows intersection set



32. The graph of R_1 in 2-space is the vertical strip of the plane between and excluding the lines $x = -4$ and $x = 4$.



The graph of R_2 in 2-space is the horizontal strip of the plane between and excluding lines $y = 4$, $y = -4$. The cross-hatch in the graph represents $R_1 \cap R_2$.

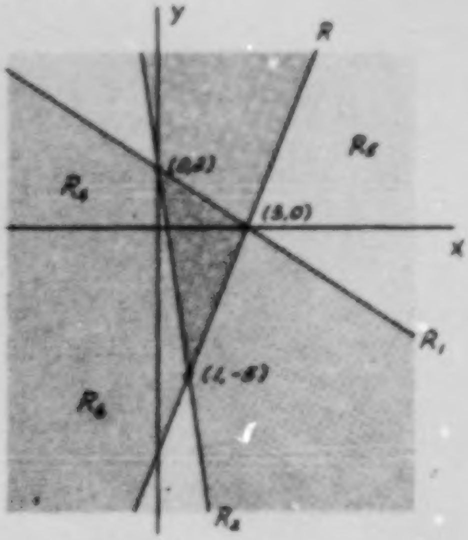
In one-space R_1 is a segment between and excluding points $x = 4$ and $x = -4$; for R_2 the same situation prevails on the y-axis. (The line for points y may be any line.) $R_1 \cap R_2$ is a single point, provided the x-axis intersects the y-axis.

In 3-space we can visualize R_1 and R_2 as the path of the 2-space graph for each separate set as it moves perpendicular to the plane of the page; $R_1 \cap R_2$ as a rectangular solid perpendicular to the plane of the page. The bounding planes are excluded from the graphs.

33. If $<$ is replaced by $\leq$ the graphs would be as in Exercise 18 except the boundaries would be included in every case. For $R_1 \cup R_2$ apply definition of union of sets. The instructor may very well use this group of exercises as an informal introduction to families of curves. Note the role of the parameter.

34. Use two-point or point-slope or otherwise to obtain $F = \frac{9}{5}C + 32$ and $C = \frac{5}{9}(F - 32)$. Science students need not memorize the formula; they can derive it.

35. The separate graphs R_1 to R_6 are labeled in the figure. $R_4 \cap R_5 \cap R_6$ is the set of all points on the triangle and its interior as shown by the cross hatch.


y
R
R5
(0,2)
R4
(3,0)
x
R1
(1,-6)
R6
R2

Challenge Exercises

1 Good students should enjoy this confrontation with ideas that go beyond the routine.

(a) Set of lines parallel to y-axis through points (x, 0) where x ranges over the integers.

(b) Set of lines parallel to the x-axis through points (0, y) where y ranges over the integers.

(c) The set of all lattice points of the plane.

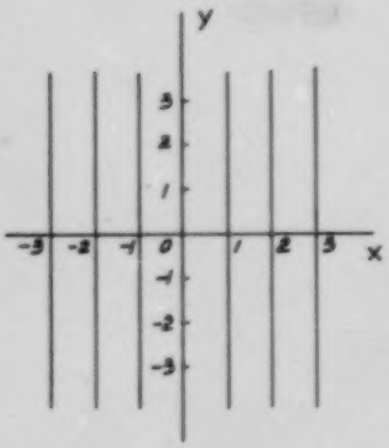
(d) Includes all of R_1, R_2, R_3. A grill such as paper ruled in cross section.

(e) Boundaries on the heavy sides are included.

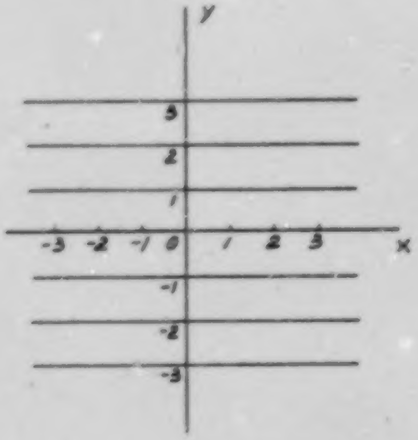
(f) Same graph moved k units to the right.

(g) and (h) Notice effect of placement of minus signs.

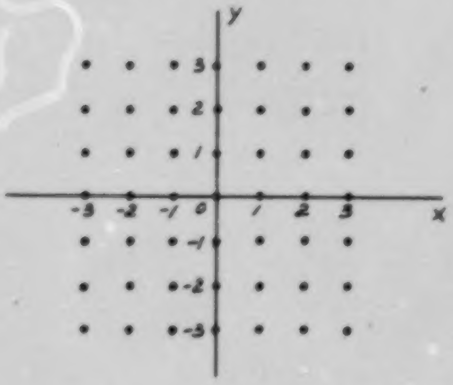
(a) $R_1 = \{(x, y):[x] = x\}$



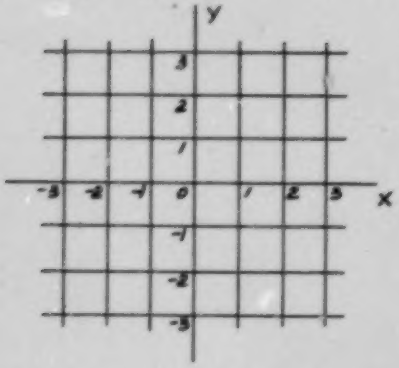
(b) $R_2 = \{(x, y):[y] = y\}$



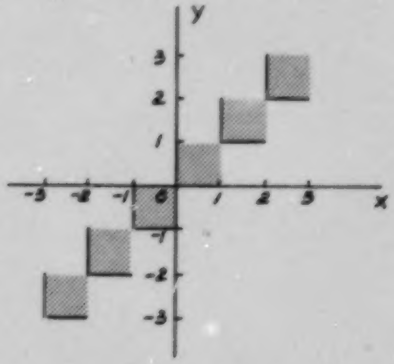
(c) $R_3 = \{(x, y):[x] = x\} \cap \{(x, y):[y] = y\}$



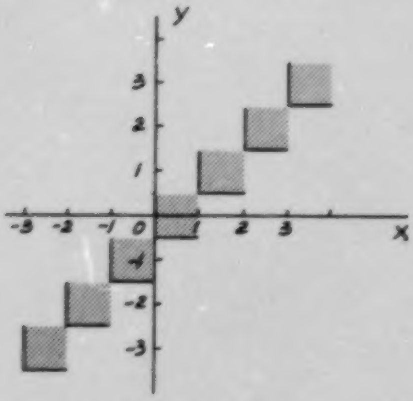
(d) $R_4 = R_1 \cup R_2$



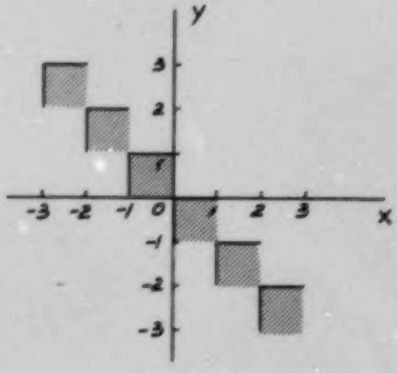
(e) $R_5 = \{(x, y):[x] = [y]\}$



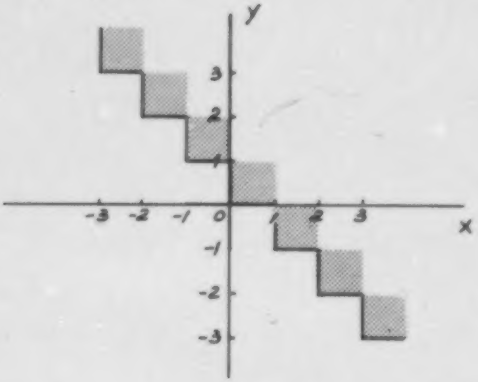
(f) $R_6 = \{(x, y):[x] = [y + k]\}$



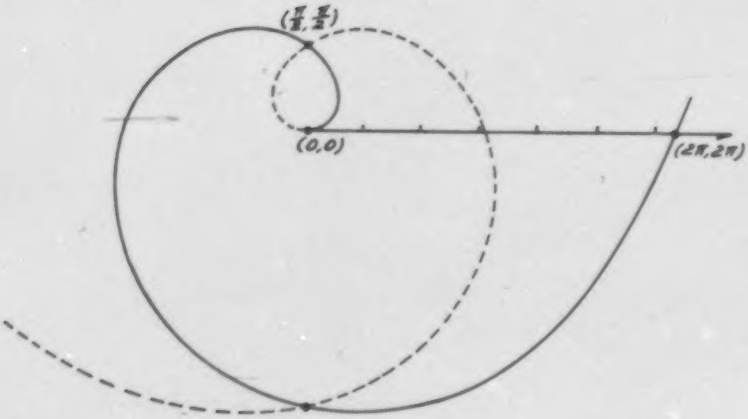
(g) $R_7 = \{(x, y):[x] = [-y]\}$



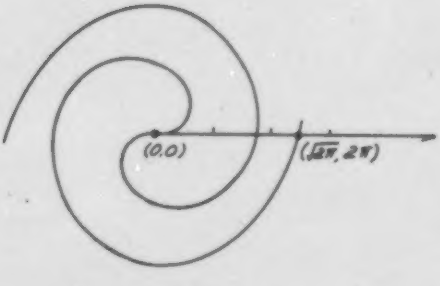
(h) $R_8 = \{(x, y):[x] = -[y]\}$



2. $\{(r, \theta):r = \theta\}$ [dotted line accounts for negative values of r]



3. $\{(r, \theta):r^2 = \theta\}$



4. (a) $d(P_1,P_2) = \sqrt{(x_2 - x_1)^2 + (\frac{r}{s})^2(y_2 - y_1)^2}$

(b) $d(P_1,P_2) = \sqrt{(\frac{s}{r})^2(x_2 - x_1)^2 + (y_2 - y_1)^2}$

(c) $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ must either be parallel or have supplementary inclinations.

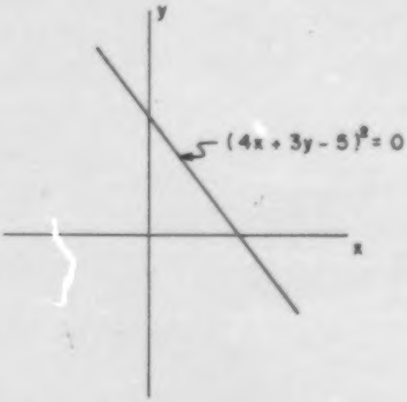
Let $\alpha = \frac{r}{s}$. From part (a) we know that for $d(P,Q) = d(R,S)$ we must have $(p_1 - q_1)^2 + \alpha^2(p_2 - q_2)^2 = (r_1 - s_1)^2 + \alpha^2(r_2 - s_2)$.

But also $(p_1 - q_1)^2 + (p_2 - q_2)^2 = (r_1 - s_1)^2 + (r_2 - s_2)^2$.

Thus $(1 - \alpha^2)(p_2 - q_2)^2 = (1 - \alpha^2)(r_2 - s_2)^2$. Since $r \neq s$, we know $\alpha^2 \neq 1$. Therefore, $1 - \alpha^2 \neq 0$ and we may divide by $1 - \alpha^2$.

From the result we see that the distances in the y-direction must be equal. But then the distances in the x-direction must be equal. These conditions are satisfied only when $\overline{PQ}$ and $\overline{RS}$ are parallel or when they have supplementary inclinations.

5.

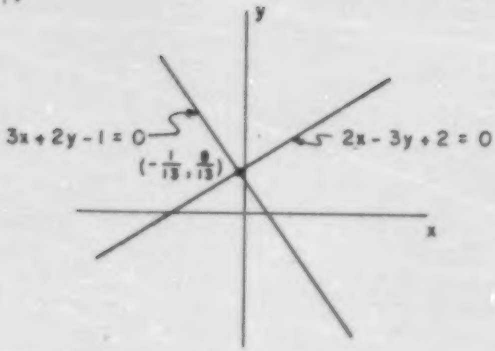


The line may be written in a simpler analytic representation.

$$4x + 3y - 5 = 0 \ .$$

6. The graph of $(ax + by + c)^k = 0$ is the same as the graph of $ax + by + c = 0$. A simpler representation is $ax + by + c = 0$.

7.


y
3x + 2y - 1 = 0
2x - 3y + 2 = 0
$\left(-\frac{1}{13}, \frac{8}{13}\right)$
x

$$3x + 2y - 1 = 0$$

$$y = -\frac{3}{2}x + \frac{1}{2}$$

$$2x - 3y + 2 = 0$$

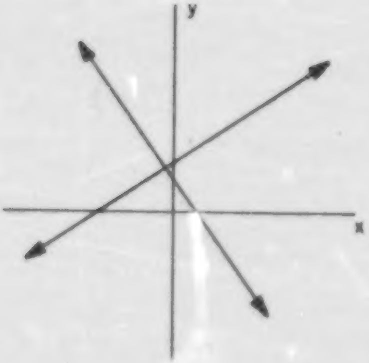
$$y = \frac{2}{3}x + \frac{2}{3}$$

$$-\frac{3}{2}x + \frac{1}{2} = \frac{2}{3}x + \frac{2}{3}$$

$$x = -\frac{1}{13}$$

$$y = \frac{8}{13}$$

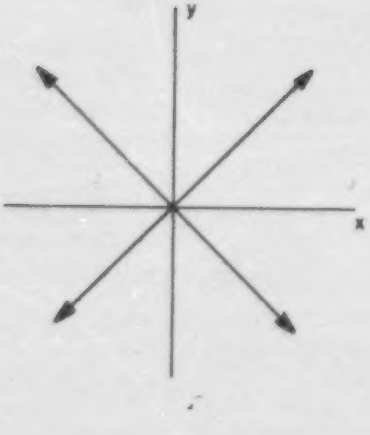
8.



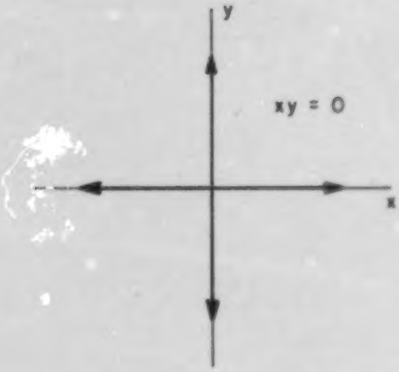
$$(3x + 2y - 1)(2x - 3y + 2) = 0$$

9.

$$(x + y)(x - y) = 0$$



10.



11. (a) rational
    (b) rational
    (c) real
    (d) complex

12. (a) R may be any line containing the point $(-\frac{4}{5}, -\frac{1}{5})$ except

$$L = \{(x,y) : x + y + 1 = 0\} .$$

(b) S may be any line containing the point $(-\frac{4}{5}, -\frac{1}{5})$ except

$$L = \{(x,y) : 3x - 2y + 2 = 0\} .$$

(c) T may be any line containing the point $(-\frac{4}{5}, -\frac{1}{5})$.

13. (a) U is the whole plane except for the points of

$$L = \{(x,y) : x + y + 1\}$$

other than $(-\frac{4}{5}, -\frac{1}{5})$.

(b) V is the whole plane except for the points of

$$L = \{(x,y) : 3x - 2y + 2 = 0\}$$

other than $(-\frac{4}{5}, -\frac{1}{5})$

(c) W is the whole plane.

14. There are two possibilities: $L_0 = \{(x,y) : a_0x + b_0y + c_0 = 0\}$ and $L_1 = \{(x,y) : a_1x + b_1y + c_1 = 0\}$ may intersect at a point (x_0,y_0) . In this case,

(a) R may be any line containing (x_0,y_0) except L_1 ,

(b) S may be any line containing (x_0,y_0) except L_0 ,

(c) T is the whole plane except those points of L_1 other than (x_0,y_0) ,

(d) U is the whole plane except those points of L_0 other than (x_0,y_0) ,

(e) V may be any line containing (x_0,y_0) , and

(f) W is the whole plane.

L_0 and L_1 may be parallel. In this case,

(a) unless R is empty, it is a line parallel to L_0 and L_1 except L_1 , when $k = 0$, $R = L_0$; when $0 < k$ R is between L_0 and L_1 ; when $-1 < k < 0$, L_0 is between L_1 and R ; when $k = -1$, R is empty (the null set); when $k < -1$, L_1 is between L_0 and R .

(b) The same argument holds for S , but the roles of L_0 and L_1 are reversed.

(c) T is the whole plane except L_1 .

(d) U is the whole plane except L_0 .

(e) unless V is empty, it is a line parallel to L_0 and L_1 . When $n = 0$, $V = L_0$; when $m = 0$, $V = L_0$.

(f) W is the whole plane.

15. (a) the null (or empty) set.

(b) the whole plane.

We include a copious set of Illustrative Test Items from which we may wish to make selections.

Illustrative Test Items for Sections 2-1 through 2-5

1. If P and Q have coordinates 3 and -5 respectively in one linear coordinate system on the line and corresponding coordinates -2 and 3 respectively in a second linear coordinate system, what are the corresponding coordinates of points with the following coordinates in the first coordinate system?

    (a) 0
    (b) 1
    (c) -1
    (d) $-\frac{1}{5}$
    (e) $-1\frac{4}{5}$
    (f) -13
    (g) 11
    (h) 10

2. If M , A , and B are the midpoint and trisection points of $\overline{PQ}$, find m , a , and b when

    (a) $p = 3$, $q = 12$
    (b) $p = -3$, $q = 1$
    (c) $p = -2$, $q = 13$
    (d) $p = 2r + 3s$, $q = 3r - 2s$

3. If the coordinates of P , Q , and R are 2 , x , and 12 respectively, find the value(s) of x such that

    (a) $d(P,Q) = \frac{1}{5} d(P,R)$
    (b) $d(P,R) = 2d(P,Q)$
    (c) $d(P,Q) = 5d(P,R)$
    (d) $d(P,Q) = 2d(R,P)$
    (e) $d(Q,P) = \frac{1}{2}d(P,R)$

4. If M , A , and B are the midpoint and trisection points of $\overline{PQ}$, find the coordinates of M , A , and B when

    (a) $P = (2,1)$, $Q = (-4,-2)$
    (b) $P = (7,1)$, $Q = (-2,1)$
    (c) $P = (-2,5)$, $Q = (7,12)$
    (d) $P = (p_1,p_2)$, $Q = (q_1,q_2)$
    (e) $P = (1,r)$, $Q = (s + r, 2s - 3)$

5. P, Q, and R are points in a plane with a rectangular coordinate system. Determine whether the three points are collinear if

(a) $P = (-5,5)$, $Q = (0,0)$, $R = (7,-7)$
(b) $P = (-1,5)$, $Q = (8,-3)$, $R = (-7,-6)$
(c) $P = (1,2)$, $Q = (9,10)$, $R = (-3,-2)$
(d) $P = (9,-10)$, $Q = (-8,5)$, $R = (0,-2)$

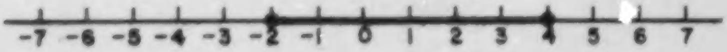
6. A line with slope $-\frac{2}{3}$ passes through $(-3,4)$. If the points $(p,7)$ and $(5,q)$ are on the line, find p and q.

7. Sketch the graphs of the sets of points on a line with the following analytic representations.

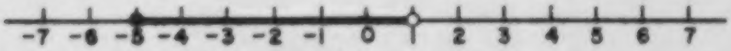
(a) $[x:\ -1 \leq x < 4]$
(b) $[x:\ |x - 5| < 2]$
(c) $[x:\ (x - 1)(x - 3) \leq 0]$
(d) $[x:\ x(x + 2)(x - 3) = 0]$

8. Find analytic conditions which describe the illustrated sets of points.

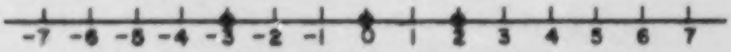
(a)


-7
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5
6
7

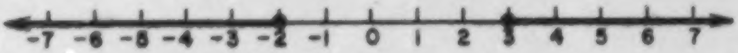
(b)


-7
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5
6
7

(c)


-7
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5
6
7

(d)


-7
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5
6
7

9. Find three polar representations for the point with rectangular coordinates

(a) $(3,3\sqrt{3})$
(b) $(-2,-2)$
(c) $(-1,\sqrt{3})$
(d) $(-2\sqrt{3},-2)$
(e) $(4,-4)$
(f) $(1,\frac{1}{\sqrt{3}})$
(g) $(6,0)$
(h) $(0,-12)$

10. Find rectangular coordinates for the point with polar coordinates

(a) $(4,0)$

(b) $(\sqrt{2},45^\circ)$

(c) $(6,-120^\circ)$

(d) $(5,\frac{5\pi}{6})$

(e) $(-3,-\frac{3\pi}{4})$

(f) $(-4,-\frac{11\pi}{6})$

11. Without changing to rectangular coordinates find the distance between the points whose polar coordinates are

(a) $(5,0)$ and $(12,\frac{\pi}{2})$

(b) $(6,0)$ and $(6,-\pi)$

(c) $(4,45^\circ)$ and $(5,-135^\circ)$

(d) $(3,\frac{\pi}{3})$ and $(4,\frac{2\pi}{3})$

(e) $(-6,-\frac{\pi}{4})$ and $(5,\frac{\pi}{4})$

(f) $(-3,-90^\circ)$ and $(6,90^\circ)$

12. Find an equation in the indicated form for the line which

(a) contains $(5,3)$ and $(6,4)$; symmetric form.

(b) contains $(0,4)$ and $(3,0)$; intercept form.

(c) contains $(7,-6)$, slope $-\frac{2}{3}$; point-slope form.

(d) contains $(13,-6)$ and $(-2,12)$; general form.

(e) contains $(0,-5)$, slope $\frac{3}{2}$; slope-intercept form.

(f) contains $(9,10)$ and $(-\sqrt{2},4)$; two-point form.

(g) contains $(-5,12)$, inclination $\frac{3\pi}{4}$; point-slope form.

(h) contains $(5,7)$ and $(5,-3)$; two-point form.

(i) contains $(3,-6)$ and $(-3,3)$; intercept form.

(j) x-intercept 2 ; y-intercept 4 ; general form.

(k) x-intercept 5 ; inclination 60° ; slope-intercept form.

(l) contains $(-5,7)$, slope $\frac{6}{7}$; symmetric form.

(m) contains $(-5,-4)$, inclination 45° ; general form.

(n) contains $(7,-2)$, slope $\frac{7}{13}$; symmetric form.

(o) contains $(6,-5)$ and $(-3,2)$; two-point form.
(p) contains $(3,4)$, slope -2 ; intercept form.
(q) contains $(6,1)$ and $(-2,5)$; slope-intercept form.
(r) contains $(9,3)$ and $(9,12)$; general form.
(s) contains $(2,3)$ and $(-7,3)$; general form.
(t) contains $(-5,4)$, inclination $\frac{2\pi}{3}$; point-slope form.

13. Show that the triangle ABC is a right triangle if $A = (-1,-3)$, $B = (11,8)$, and $C = (-3,4)$.

14. Find an equation in general form of the line containing the median to side $\overline{BC}$ of triangle ABC if $A = (-2,7)$, $B = (3,4)$, and $C = (1,-2)$.

15. Find the area of the triangle determined by the lines

$$L_1 = \{(x,y): \ 2x - 8 = 0\} ,$$
$$L_2 = \{(x,y): \ 12x - 5y - 53 = 0\} ,$$
$$L_3 = \{(x,y): \ 4x - 5y + 19 = 0\} .$$

16. In triangle ABC , $A = (0,0)$, $B = (6,0)$ and $C = (0,8)$.
(a) The bisector $\angle A$ divides the segment $\overline{BC}$ in what ratio?
(b) The point D at which the bisector of $\angle A$ intersects $\overline{BC}$?
(c) Find $d(B,D)$ and $d(C,D)$.

17. Find the coordinates of the points in which the line that contains $(-8,3)$ and $(3,-2)$ intersects the axes.

Answers

1. (a) $-\frac{1}{8}$
(b) $-\frac{3}{4}$
(c) $\frac{1}{2}$
(d) 0
(e) 1
(f) 8
(g) -7
(h) $-6\frac{3}{8}$

2. (a) $m = 7\frac{1}{2}$, $a = 6$, $b = 9$

   (b) $m = -1$, $a = -1\frac{2}{3}$, $b = -\frac{1}{3}$

   (c) $m = 5\frac{1}{2}$, $a = 3$, $b = 8$

   (d) $m = \frac{5r + s}{2}$, $a = \frac{7r + 4s}{3}$, $b = \frac{8r - s}{3}$

3. (a) 0 , 4
   (b) -3 , 7
   (c) -48 , 52
   (d) -18 , 22
   (e) -3 , 7

4. (a) $M = (-1, -\frac{1}{2})$ $A = (0,0)$ $B = (-2,-1)$

   (b) $M = (2\frac{1}{2}, 1)$ $A = (4,1)$ $B = (1,1)$

   (c) $M = (2\frac{1}{2}, 8\frac{1}{2})$ $A = (1, 7\frac{1}{3})$ $B = (4, 9\frac{2}{3})$

   (d) $M = \left(\frac{p_1 + q_1}{2}, \frac{p_2 + q_2}{2}\right)$ $A = \left(\frac{2p_1 + q_1}{3}, \frac{2p_2 + q_2}{3}\right)$ $B = \left(\frac{p_1 + 2q_1}{3}, \frac{p_2 + 2q_2}{3}\right)$

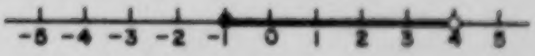
   (e) $M = \left(\frac{r + s + 1}{2}, \frac{r + 2s - 3}{2}\right)$ $A = \left(\frac{r + s + 2}{3}, \frac{2r + 2s - 3}{3}\right)$ $B = \left(\frac{2r + 2s + 1}{3}, \frac{r + 4s - 6}{3}\right)$

5. (a) Yes
   (b) No
   (c) Yes
   (d) No

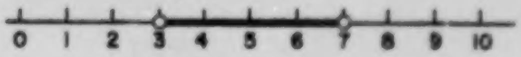
(Determine the distances between the pairs of points; the points are collinear if and only if the sum of the two shorter distances equals the longer. More simply, use slopes; the points are collinear if and only if the slope of $\overline{PQ}$ equals the slope of $\overline{PR}$.)

6. $p = -7\frac{1}{2}$, $q = -\frac{4}{3}$.

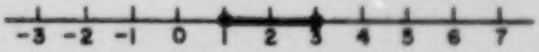
7. (a)



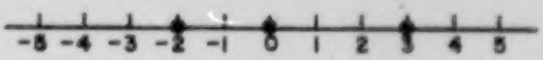
(b)



(c)



(d)



8. (a) $\{x:\ -2 \le x \le 4\}$, $\{x:\ |x - 1| \le 3\}$, $\{x:\ (x + 2)(x - 4) \le 0\}$, or the equivalent.

(b) $\{x:\ -5 \le x < 1\}$, $\{x:\ \frac{x - 1}{x - 1}|x + 2| \le 3\}$, $\{x:\ \frac{x - 1}{x - 1}(x + 5)(x - 1) \le 0\}$, or the equivalent.

(c) $\{x:\ x(x + 3)(x - 2) = 0\}$, $\{-3, 0, 2\{$, or the equivalent.

(d) $\{x:\ x \le -2$ or $x \ge 3\}$, $\{x:\ |x - \frac{1}{2}| \ge 2\frac{1}{2}\}$, $\{x:\ (x + 2)(x - 3) \ge 0\}$, or the equivalent.

9. (There are, of course, unlimited possibilities for the answers to this question; we give only a few.)

(a) $(6,\frac{\pi}{3})$, $(-6,\frac{4\pi}{3})$, $(6,60^\circ)$, $(-6,240^\circ)$.

(b) $(2\sqrt{2},\frac{5\pi}{4})$, $(-2\sqrt{2},\frac{\pi}{4})$, $(2\sqrt{2},225^\circ)$, $(-2\sqrt{2},45^\circ)$

(c) $(2,\frac{2\pi}{3})$, $(-2,\frac{5\pi}{3})$, $(2,120^\circ)$, $(-2,300^\circ)$

(d) $(4,\frac{7\pi}{6})$, $(-4,\frac{\pi}{6})$, $(4,210^\circ)$, $(-4,30^\circ)$

(e) $(4\sqrt{2},\frac{7\pi}{4})$, $(-4\sqrt{2},\frac{3\pi}{4})$, $(4\sqrt{2},315^\circ)$, $(-4\sqrt{2},135^\circ)$

(f) $(\frac{2}{\sqrt{3}},\frac{\pi}{6})$, $(-\frac{2}{\sqrt{3}},\frac{7\pi}{6})$, $(\frac{2}{\sqrt{3}},30^\circ)$, $(-\frac{2}{\sqrt{3}},210^\circ)$

(g) $(6,0)$, $(-6,\pi)$, $(6,0^\circ)$, $(-6,180^\circ)$

(h) $(12,\frac{3\pi}{2})$, $(-12,\frac{\pi}{2})$, $(12,270^\circ)$, $(-12,90^\circ)$

10. (a) $(4,0)$
(b) $(1,1)$
(c) $(-3,-3\sqrt{3})$
(d) $(-\frac{5\sqrt{3}}{2}, 2\frac{1}{2})$
(e) $(\frac{3}{\sqrt{2}},\frac{3}{\sqrt{2}})$
(f) $(-2\sqrt{3},2)$

11. (a) 13
(b) 12
(c) 9
(d) $\sqrt{13}$
(e) $\sqrt{61}$
(f) 3

12. (a) $\frac{x - 5}{6 - 5} = \frac{y - 3}{4 - 3}$
(b) $\frac{x}{3} + \frac{y}{4} = 1$
(c) $y + 6 = -\frac{2}{3}(x - 7)$
(d) $6x + 5y - 48 = 0$
(e) $y = \frac{3}{2}x - 5$
(f) $y - 10 = \frac{4 - 10}{-\sqrt{2} - 9}(x - 9)$
(g) $y - 12 = -1(x + 5)$
(h) $x - 5 = \frac{5 - 5}{-3 - 7}(y - 7)$
(i) $\frac{x}{-1} + \frac{y}{-\frac{3}{2}} = 1$
(j) $4x - 2y - 8 = 0$
(k) $y = \sqrt{3}\,x - 5\sqrt{3}$
(l) $\frac{x + 5}{2 + 5} = \frac{y - 7}{13 - 7}$
(m) $x - y + 1 = 0$
(n) $\frac{x - 7}{20 - 7} = \frac{y + 2}{5 + 2}$
(o) $y + 5 = \frac{2 + 5}{-3 - 6}(x - 6)$
(p) $\frac{x}{5} + \frac{y}{10} = 1$
(q) $y = -\frac{1}{2}x + 4$
(r) $x = 9$
(s) $y = 3$
(t) $y - 4 = -\sqrt{3}(x + 5)$

13. (a) $\left(d(A,B)\right)^2 = (-1 - 11)^2 + (-3 - 8)^2 = 265$

$\left(d(B,C)\right)^2 = (11 + 3)^2 + (8 - 4)^2 = 212$

$\left(d(A,C)\right)^2 = (-1 + 3)^2 + (-3 - 4)^2 = 53$

Since $\left(d(A,B)\right)^2 = \left(d(B,C)\right)^2 + \left(d(A,C)\right)^2$, by the converse of the Pythagorean Theorem triangle ABC is a right triangle with $\angle ACB$ the right angle.

(b) If you permit students to use the fact that the product of the slopes is -1 if and only if lines are perpendicular, the proof follows more readily from the fact that :

$$m_{\overline{AC}} \cdot m_{\overline{BC}} = (-\tfrac{7}{2})\cdot(\tfrac{2}{7}) = -1 \; .$$

14. $3x + 2y - 8 = 0$

15. 20

16. (a) 3 to 4

(b) $(3\tfrac{3}{7}, 3\tfrac{3}{7})$

(c) $d(B,D) = 4\tfrac{2}{7}$; $d(C,D) = 5\tfrac{5}{7}$

17. The line intersects the x-axis at $(-\tfrac{7}{5}, 0)$;

the line intersects the y-axis at $(0, -\tfrac{7}{11})$.

57-63 Most students will probably believe they have a clear intuitive understanding of the idea of the two directions on a line and may feel the discussion here is pointless. As with the notion of a directed segment, it may help to ask them to try to explain what they mean accurately, using terms with clear geometric meanings. When they find that this is not at all easy, they may be convinced that our approach is worth studying.

57 The open question of lines without slope is considered in Exercise 5 on page 64. At this point we assume that the student recalls that parallel, nonvertical lines have the same slope. In Section 2-7 we shall reafirm this fact.

57 We shall use the idea of equivalent direction numbers for a line a great deal; if a student does not grasp this idea now, he may find it a frequent stumbling block.

58 You may well note that had we chosen directed angles to describe the lines in the plane, a single angle would suffice. However, a pair of nonnegative angles is conventional and leads to symmetric representation; it is also desirable, since a triple of direction angles is much neater in 3-space. The extension to spaces of higher dimension is immediate with the approach adopted here.

59-60 The fact that the pair of normalized direction numbers and the pair of direction cosines are equal is extremely convenient.

61 The context which specifies a direction for a line varies and is, of course, frequently quite colloquial, as "the line from P to Q".

Exercise 6 on page 64 asks for a justification that the alternative direction angles for a line are respectively supplementary.

62 The information developed in the solution to Example 4(b) is quite useful. The student should develop facility in extracting from a general form of an equation of a line direction numbers and direction cosines for the line.

63 The importance of Example 5 may not be apparent. It provides what little initfal motivation there is for the normal form of the equation of a line.

64 Exercise 7 might well be discussed briefly even if it is not assigned, for it develops a relationship which is useful in relating the equations of a line in polar and rectangular coordinates.

Exercises 2-6

1. (a) $(-3,4)$ or $(3,-4)$
   (b) $(4,1)$ or $(-4,-1)$
   (c) $(0,6)$ or $(0,-6)$
   (d) $(-5,0)$ or $(5,0)$
   (e) $(1,1)$ or $(-1,-1)$
   (f) $(2,2)$ or $(-2,-2)$
   (g) $(-1,1)$ or $(1,-1)$
   (h) $(-4,4)$ or $(4,-4)$

2. (a) $(-\frac{3}{5}, \frac{4}{5})$ or $(\frac{3}{5}, -\frac{4}{5})$

   (b) $(\frac{4}{\sqrt{17}}, \frac{1}{\sqrt{17}})$ or $(\frac{-4}{\sqrt{17}}, \frac{-1}{\sqrt{17}})$

   (c) $(0,1)$ or $(0,-1)$

   (d) $(-1,0)$ or $(1,0)$

   (e) $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ or $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$

   (f) $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ or $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$

   (g) $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ or $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$

   (h) $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ or $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$

3. (a) $\alpha = 127^\circ$, $\beta = 37^\circ$; or $\alpha = 53^\circ$, $\beta = 143^\circ$ (approximately)

   (b) $\alpha = 76^\circ$, $\beta = 14^\circ$; or $\alpha = 104^\circ$, $\beta = 166^\circ$ (approximately)

   (c) $\alpha = 90^\circ$, $\beta = 0^\circ$; or $\alpha = 90^\circ$, $\beta = 180^\circ$

   (d) $\alpha = 180^\circ$, $\beta = 90^\circ$; or $\alpha = 0^\circ$, $\beta = 90^\circ$

   (e) $\alpha = 45^\circ$, $\beta = 45^\circ$; or $\alpha = 135^\circ$, $\beta = 135^\circ$

   (f) $\alpha = 45^\circ$, $\beta = 45^\circ$; or $\alpha = 135^\circ$, $\beta = 135^\circ$

   (g) $\alpha = 135^\circ$, $\beta = 45^\circ$; or $\alpha = 45^\circ$, $\beta = 135^\circ$

   (h) $\alpha = 135^\circ$, $\beta = 45^\circ$; or $\alpha = 45^\circ$, $\beta = 135^\circ$

4. (a)

$(3,-4)$	$(2,0)$	$(0,-3)$	$(-1,2)$	$(-2,1)$
$-\frac{4}{3}$	0	not defined	-2	$-\frac{1}{2}$

   (b) $(\frac{3}{5}, -\frac{4}{5})$, or any equivalent given by $(\frac{3c}{5}, -\frac{4c}{5})$, $c \neq 0$.

   $\alpha = 53^\circ$, $\beta = 143^\circ$; or $\alpha = 127^\circ$, $\beta = 37^\circ$

   $(1,0)$, or any equivalent given by $(c,0)$, $c \neq 0$.

   $\alpha = 0^\circ$, $\beta = 90^\circ$; or $\alpha = 180^\circ$, $\beta = 90^\circ$

   $(0,-1)$, or any equivalent given by $(0,-c)$, $c \neq 0$.

   [illegible] 90°, $\beta = 180^\circ$; or $\alpha = 90^\circ$, $\beta = 0^\circ$

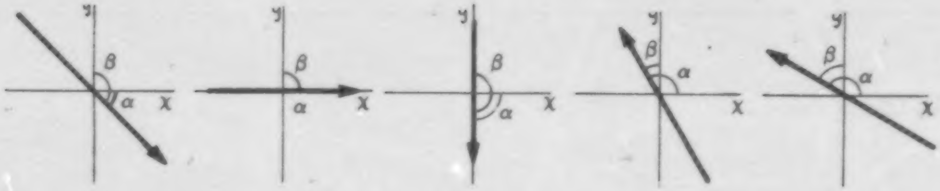
   $(-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}})$, or any equivalent given by $(-\frac{c}{\sqrt{5}}, \frac{2c}{\sqrt{5}})$, $c \neq 0$.

   $\alpha = 117^\circ$, $\beta = 27^\circ$; or $\alpha = 63^\circ$, $\beta = 153^\circ$

   $(-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}})$, or any equivalent given by $(-\frac{2c}{\sqrt{5}}, \frac{c}{\sqrt{5}})$, $c \neq 0$.

   $\alpha = 153^\circ$, $\beta = 63^\circ$; or $\alpha = 27^\circ$, $\beta = 117^\circ$

(c) and (d)



5. A pair of direction numbers determined by P_0 and P_1 are

$(\ell_1, m_1) = (0, y_1 - y_0)$; $m_1 = y_1 - y_0 \neq 0$, $\ell_1 = 0$.

A pair of direction numbers determined by P_0 and P_2 are

$(\ell_2, m_2) = (0, y_2 - y_0)$; $m_2 = y_2 - y_0 \neq 0$ and $\ell_2 = 0$.

Since $m_1 \neq 0$ and $m_2 \neq 0$, both

$$c_1 = \frac{m_2}{m_1} \quad \text{and} \quad c_2 = \frac{m_1}{m_2}$$

are defined and not equal to zero. Thus,

$$(c_1\ell_1 , c_1m_1) = (\ell_2 , m_2) \quad \text{and} \quad (c_2\ell_2 , c_2m_2) = (\ell_1 , m_1) .$$

$$(0, y_1 - y_0) \quad \text{and} \quad (0, y_2 - y_0)$$

are equivalent pairs of direction numbers for the vertical line.

6. $\cos \alpha = \dfrac{\ell}{\sqrt{\ell^2 + m^2}}$ $\qquad$ $\cos \beta = \dfrac{m}{\sqrt{\ell^2 + m^2}}$

$\cos \alpha' = \dfrac{\ell}{\sqrt{\ell^2 + m^2}}$ $\qquad$ $\cos \beta' = \dfrac{-m}{\sqrt{\ell^2 + m^2}}$

So $\cos \alpha' = - \cos \alpha$ $\qquad$ $\cos \beta' = - \cos \beta$

Hence $\alpha' = \pm \alpha + p\pi$ $\quad$ $\beta' = \pm \beta + q\pi$ $\quad$ p, q odd integers but α, α', β, and β' are between 0 and π, so the only solutions are $\beta' + \beta = \pi$, $\alpha' + \alpha = \pi$.

7. (a) 1. In the Figure 2-13a , $\omega = \frac{\pi}{2} - \beta + 2\pi n$. Therefore

$\sin \omega = \sin (\frac{\pi}{2} - \beta)$ but since the sine of an angle is equal to the cosine of its complement,

$$\sin \omega = \cos \beta$$

2. In Figure 2-13b , $\omega = \beta - \frac{\pi}{2} + 2\pi n$. Therefore

$$\sin \omega = \sin (\beta - \frac{\pi}{2})$$

$$\sin \omega = \sin [-(\frac{\pi}{2} - \beta)]$$

$$\sin \omega = \cos (-\beta)$$

$$\sin \omega = \cos \beta$$

3. In Figure 2-13c, $\omega + \frac{\pi}{2} + \beta = 180 + 2\pi n$ and $\omega = \frac{\pi}{2} - \beta + 2\pi n$.

The result is the same as part 1 above.

4. In Figure 2-13d, $\omega - \beta = \frac{\pi}{2} + 2\pi n$ and $\omega = \frac{\pi}{2} + \beta + 2\pi n$.

Therefore $\sin \omega = \sin (\frac{\pi}{2} + \beta)$

$$\sin \omega = \sin \frac{\pi}{2} \cos \beta + \cos \frac{\pi}{2} \sin \beta$$

Since $\sin \frac{\pi}{2} = 1$ and $\cos \frac{\pi}{2} = 0$

$$\sin \omega = \cos \beta$$

(b) 1. If the positive ray lies on the positive half of the x-axis,

$$\omega = 2\pi n \quad \text{and} \quad \beta = \frac{\pi}{2}$$

Since we wish to show that $\sin \omega = \cos \beta$, we may substitute and see that

$$\sin 2\pi n = \cos \frac{\pi}{2} = 0$$

2. If the positive ray lies on the positive half of the y-axis,

$$\omega = \frac{\pi}{2} + 2\pi n \quad \text{and} \quad \beta = 0 \quad \text{and} \quad \sin\frac{\pi}{2} = \cos 0 = 1$$

3. If the positive ray lies on the negative half of the x-axis,

$$\omega = \pi + 2\pi n \quad \text{and} \quad \beta = \frac{\pi}{2} \quad \text{and} \quad \sin \pi = \cos\frac{\pi}{2} = 0$$

4. If the positive ray lies on the negative half of the y-axis,

$$\omega = \frac{3\pi}{2} + 2\pi n\ , \ \beta = \pi \quad \text{and} \quad \sin\frac{3\pi}{2} = \cos \pi = -1$$

8. (a) $(-2,2)$

$\left(\frac{-2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right)$

$\alpha = 153^\circ \quad \beta = 117^\circ$

(b) $(-2,1)$

$\left(\frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$

$\alpha = 153^\circ\ , \beta = 63^\circ$

(c) $(6,5)$

$\left(\frac{6}{\sqrt{61}}, \frac{5}{\sqrt{61}}\right)$

$\alpha = 40^\circ\ , \beta = 50^\circ$

65 It is traditional to talk about *the* angle between two lines, but present standards of precision require that we take account of the fact that at least four angles are formed when two lines intersect. These angles can be distinguished in a diagram by various methods, but all of these methods must induce a sense along each of the lines.

67-68 The second solution to Example (2) is given as a suggestion to the student that once he has recognized the form of the equation of the lines normal to a given line, he may write immediately the equation of the normal containing a given point.

68 Sometimes the results of our analytic approach describe additional situations not usually approached in the same way geometrically. The situation here furnishes a nice example of this.

69 Example 3(b) is also offered to show the student how he may use an equation of a given line in general form to write immediately an equation of a parallel line containing a given point.

70-71 Since $(b_1,-a_1)$ and $(b_2,-a_2)$ are pairs of direction numbers for the lines L_1 and L_2 respectively, we also note that

$$\cos \theta = \frac{\ell_1\ell_2 + m_1m_2}{\ell_2^2 + m_2^2 \sqrt{\ell_2^2 + m_2^2}}$$

or

$$\cos \theta = \lambda_1\lambda_2 + \mu_1\mu_2 .$$

In Exercise 12 on page 74 the student is asked to develop this relationship. It has some merit when the lines forming $\angle \theta$ are directed lines. In this case $\angle \theta$ is the angle formed by positive rays of L_1 and L_2 with endpoints at the point of intersection (if any) of L_1 and L_2. Exercise 15 on page 87 also calls for such an interpretation.

71-72 Example 5 is really a lemma to be used in the development of the normal form of an equation of a line in the following section.

Exercises 2-7

1. (a) $d(A,C) = d(B,C) + d(A,B)$, by the definition of betweenness for points. This is equivalent to

$$d(A,B) = d(A,C) - d(B,C) ,$$

which implies

$$\left(d(A,B)\right)^2 = \left(d(A,C)\right)^2 + \left(d(B,C)\right)^2 - 2d(A,C)\, d(B,C) ;$$

since $\cos C = \cos 0^\circ = 1$, we may write

$$\left(d(A,B)\right)^2 = \left(d(A,C)\right)^2 + \left(d(B,C)\right)^2 - 2d(A,C)\, d(B,C) \cos C$$

(b) Here we have

$$d(A,B) = d(A,C) + d(B,C) ,$$

which implies

$$\left(d(A,B)\right)^2 = \left(d(A,C)\right)^2 + \left(d(B,C)\right)^2 + 2d(A,C)\,d(B,C) ;$$

since $\cos C = \cos 180^\circ = -1$, we may write

$$\left(d(A,B)\right)^2 = \left(d(A,C)\right)^2 + \left(d(B,C)\right)^2 - 2d(A,C)\,d(B,C)\cos C .$$

2. (a) Equation (6) states that

$$\cos\theta = \frac{a_1a_2 + b_1b_2}{\sqrt{a_1^2 + b_1^2}\,\sqrt{a_2^2 + b_2^2}} .$$

Substituting into Equation (6) ,

$$\cos\theta = \frac{a_1a_2}{\sqrt{a_1^2}\,\sqrt{a_2^2 + b_2^2}}$$

$$\cos\theta = \frac{\pm a_2}{\sqrt{a_2^2 + b_2^2}} .$$

Let α be the inclination of L_2 . Then the measures of the angles θ between L_1 and L_2 are $90^\circ - \alpha$ and $90^\circ + \alpha$.

$\cos\theta = \cos(90^\circ - \alpha) = \cos 90^\circ \cos\alpha + \sin 90^\circ \sin\alpha = \sin\alpha$

or

$\cos\theta = \cos(90^\circ + \alpha) = \cos 90^\circ \cos\alpha - \sin 90^\circ \sin\alpha = -\sin\alpha$.

Also we have $\tan\alpha = \dfrac{\sin\alpha}{\cos\alpha} = -\dfrac{a_2}{b_2}$

$$b_2 \sin\alpha = -a_2 \cos\alpha ,$$

and $b_2^2 \sin^2\alpha = a_2^2 \cos^2\alpha = a_2^2(1 - \sin^2\alpha)$.

This is equivalent to $\sin^2\alpha = \dfrac{a_2^2}{a_2^2 + b_2^2}$,

and

$$\sin\alpha = \frac{\pm a_2}{\sqrt{a_2^2 + b_2^2}} = \cos\theta$$

(b) $$\cos\theta = \frac{a_1 a_2}{\sqrt{a_1^2}\sqrt{a_2^2}} = \pm 1$$

$\theta = 0^\circ$ or 180° , which is the case for parallel lines.

3. L_2 and L_5 are the same lines

   L_1 and L_4 are the same lines

   L_3 is perpendicular to L_1 and L_4

4. (a) $\theta = 7^\circ$
   (b) $\theta = 90^\circ$
   (c) $\theta = 45^\circ$
   (d) $\theta = 83^\circ$
   (e) $\theta = 0^\circ$ (lines are parallel).
   (f) $\theta = 90^\circ$

5. The slope of OP is $\frac{b}{a}$ and the slope of OQ is $\frac{a}{-b}$.

   Since $m_{\overline{OP}} \cdot m_{\overline{OQ}} = -1$, $\overline{OP} \perp \overline{OQ}$.

6. (a) $2x - 3y = 0$
   (b) $3x + y - 8 = 0$
   (c) $3x + 2y - 17 = 0$
   (d) $x - 3y - 5 = 0$

7. (a) $2x - 5y + 31 = 0$
   (b) $2x - 3y + 17 = 0$
   (c) $16x - 6y - 13 = 0$
   (d) $y = 7$
   (e) $x = 5$

8. $D = (4,-8)$.

   3 possibilities; (12,2) and (-2,12) are the others.

9. The slope of L_1 is $\frac{4}{3}$

$$y + 2 = \frac{4}{3}(x - 1)$$

10. (a) $\overleftrightarrow{AB}$: $2x + 7y - 17 = 0$

$\overleftrightarrow{BC}$: $x + y - 1 = 0$

$\overleftrightarrow{CA}$: $3x + 8y - 23 = 0$

(b) $m_{\overleftrightarrow{AB}} = -\frac{2}{7}$

$m_{\overleftrightarrow{BC}} = -1$

$m_{\overleftrightarrow{CA}} = -\frac{3}{8}$

(c)

$$m\angle CBA = 151^\circ$$

$$\cos\theta_1 = \frac{2 + 7}{\sqrt{1 + 1}\,\sqrt{4 + 49}} = .874$$

$$\theta_1 = 29^\circ$$

The angle desired is the supplement of θ_1 or $180^\circ - 29^\circ$ or 151°

$$m\angle BCA$$

$$\cos\theta_2 = \frac{3 + 8}{\sqrt{1 + 1}\,\sqrt{9 + 64}} = .910^+$$

$$\theta_2 = 24^\circ$$

$$m\angle CAB =$$

$$\cos\theta_3 = \frac{6 + 56}{\sqrt{4 + 49}\,\sqrt{9 + 64}} = .997^-$$

$$\theta_3 = 5^\circ$$

(d) Altitude to side $\overleftrightarrow{AB}$

$7x - 2y + 29 = 0$

Altitude to side $\overleftrightarrow{BC}$

$x - y - 4 = 0$

Altitude to side $\overleftrightarrow{AC}$

$8x - 3y + 25 = 0$

11. (a) $L_1' = \{(x,y):\ b_1x - a_1y = 0\}$

$L_2' = \{(x,y):\ b_2x - a_2y = 0\}$

(b) $\therefore \cos\theta = \dfrac{a_1a_2 + b_1b_2}{\sqrt{a_1^2 + b_1^2}\,\sqrt{a_2^2 + b_2^2}}$ and using Equation (6),

$$\cos \phi = \frac{b_1 b_2 + a_1 a_2}{\sqrt{(b_1)^2 + (-a)^2}\,\sqrt{b_2^2 + (-a_2)^2}}$$

$$\cos \theta = \cos \phi$$

If L_1' is $\perp$ to L_1 and L_2' is $\perp$ to L_2, then the measure of an angle between L_1 and L_2 is equal to the measure of an angle between L_1' and L_2'.

12. (a) $L_1 = \{(x,y)\colon\ \lambda_1 x + \mu_1 y + c_1 = 0\}$

$L_2 = \{(x,y)\colon\ \lambda_2 x + \mu_2 y + c_2 = 0\}$

$$\cos \theta = \frac{\lambda_1\lambda_2 + \mu_1\mu_2}{\sqrt{\lambda_1^2 + \mu_1^2}\,\sqrt{\lambda_2^2 + \mu_2^2}}$$

but $$\lambda_1^2 + \mu_1^2 = \lambda_2^2 + \mu_2^2 = 1$$

$$\cos \theta = \lambda_1\lambda_2 + \mu_1\mu_2$$

(b) If $\cos \theta$ is positive $0^\circ \leq \theta \leq 90^\circ$ and $\angle\theta$ is the least angle formed by L_1 and L_2.

(c) Assume $L_1 \perp L_2$

$$m_1 = -\frac{\lambda_1}{\mu_1} \text{ and } m_2 = -\frac{\lambda_2}{\mu_2} \text{ and}$$

$$m_1 m_2 = -1 .$$

So $\left(-\frac{\lambda_1}{\mu_2}\right)\left(-\frac{\lambda_2}{\mu_2}\right) = -1$ and

$$\lambda_1\lambda_2 = -\mu_1\mu_2 \qquad \text{or}$$

$$\lambda_1\lambda_2 = -\mu_1\mu_2 = 0$$

Conversely assume $\lambda_1\lambda_2 + \mu_1\mu_2 = 0$

but $$\lambda_1\lambda_2 + \mu_2\mu_2 = 0 = \cos \theta$$

and $$\cos \theta = 0$$

$$\therefore \theta = 90^\circ \text{ and}$$

$$L_1 \perp L_2 .$$

75-78 The normal form of an equation of a line is troublesome to develop, for students have usually not considered the characterization of a line by a normal segment from the origin. Therefore, the argument for bothering to develop it at all must rest upon its applications; it is not at all a natural extension in the students' eyes. With this in mind, before beginning this section it might be helpful to challenge the students to find the distance between a line and a point not on the line. Once they have been forced to the trouble of finding (a) the slope of the perpendiculars to the given line, (b) an equation of the perpendicular containing the given point, (c) the point of intersection of this perpendicular and the given line, and (d) the distance between the point of intersection and the given point, they may be more in a mood to pursue a development which solves this problem more easily.

76 The conventional notation does lead to confusion here. It is easy for the student to confuse the coefficients in the normal form with the direction cosines of the line itself. Emphasis on the reason for the name "normal form" may shorten the period of confusion. Then, too, an oral drill on the following information to be gleaned from the normal form may help.

If $\lambda > 0$ and $\mu > 0$, the line extends above the origin from upper left to lower right; if $\lambda < 0$ and $\mu > 0$, above the origin from lower left to upper right; if $\lambda < 0$ and $\mu < 0$, below the origin from upper left to lower right; if $\lambda > 0$ and $\mu < 0$, below the origin from lower left to upper right. If $\lambda = 0$ and $\mu = 1$, the line is horizontal and above the origin; if $\lambda = 0$ and $\mu = -1$, horizontal and below the origin; if $\mu = 0$ and $\lambda = 1$, vertical and to the right of the origin; if $\mu = 0$ and $\lambda = -1$, vertical and to the left of the origin.

To make sense of this information a student will have to keep in mind that (λ, μ) is the pair of direction cosines of the normal segment.

77 The fact that authorities differ in the case of lines containing the origin has a backhanded sort of significance. There seems to be little reason to recognize a difference which does not make a difference. E.g., $1.\overline{0} = 0.\overline{9}$; there is no numerical difference.

78 If your students are already versed in the parametric representation of lines, there is a neater approach to the problem.

The line $\overleftrightarrow{FP_1}$ has the parametric representation

$$x = x_1 + \lambda t$$

$$y = y_1 + \mu t .$$

With this representation $|t|$ is the distance between (x,y) and $P_1 = (x_1,y_1)$. In particular, if we let $F = (x_0,y_0)$, for some t, F has a representation

$$x_0 = x_1 + \lambda t_1$$
$$y_0 = y_1 + \mu t_1 .$$

Then

$$x_0 - x_1 = \lambda t_1$$
$$y_0 - y_1 = \mu t_1 ,$$

and

$$d(P_1,F) = \sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2} = \sqrt{(\lambda t_1)^2 + (\mu t_1)^2}$$
$$= |t_1| \sqrt{\lambda^2 + \mu^2} = |t_1| .$$

Since the point $F = (x_0,y_0)$ satisfies the equation $\lambda x + \mu y - p = 0$, we have

$$\lambda(x_1 + \lambda t_1) + \mu(y_1 + \mu t_1) - p = 0 ,$$

which is equivalent to

$$\lambda x_1 + \mu y_1 - p = -(\lambda^2 + \mu^2)\, t_1 = -t_1 .$$

Thus,

$$d(P_1,F) = |t_1| = |\lambda x_1 + \mu y_1 - p| .$$

With this approach we do not have to consider the five different cases.

79-82 The amount of classroom explication necessary on the polar form will depend upon the students' background in analytic trigonometry. Some familiarity with the addition formulas is essential. These are developed in SMSG Intermediate Mathematics, pages 605-610, and, of course, in any standard trigonometry text.

79 At this point you may wish to consider that since $P = (-r,\theta + \pi)$, the line also has the polar representation

$$-r \cos\left(\theta + (\pi - \omega)\right) = p .$$

This opens a question to which we shall return in Chapter 5, when we consider related polar equations.

80 Although the polar angle which contains the normal segment to L is the same set of points as the direction angle α and $\angle\omega = \angle\alpha$, our conventions for measuring these angles are different. The measure of $\angle\omega$

may be any real number, while $0 \le \alpha \le \pi$ (or $0 \le \alpha \le 180^\circ$) . Thus, even if we choose an $\angle\omega$ such that $|\omega|$ is minimal, we still are assured only that $|\omega| = \alpha$, or $\omega = \pm \alpha$. However, since $\omega = \pm \alpha + 2\pi n$ for any integer n, in the case we describe, we do have $\cos \omega = \cos (2\pi n \pm \alpha) = \cos \alpha$. The test should read $\omega = \pm \alpha + 2\pi n$ for any integer n .

81 Students may not be familiar with the technique of "normalizing" coefficients in order to rewrite

$$a \cos \theta + b \sin \theta \text{ as } \sqrt{a^2 + b^2} \sin(\theta + \alpha_1) .$$

where

$$\sin \alpha_1 = \frac{a}{\sqrt{a^2 + b^2}} \text{ and } \cos \alpha_1 = \frac{b}{\sqrt{a^2 + b^2}} ,$$

or as $\sqrt{a^2 + b^2} \cos (\theta - \beta_1)$, where

$$\cos \beta_1 = \frac{a}{\sqrt{a^2 + b^2}} \text{ and } \sin \beta_1 = \frac{b}{\sqrt{a^2 + b^2}} .$$

Therefore, you may wish to consider other examples than Part (e) of Example 5.

84-85 In assigning exercises you may well wish to consider Exercises 7 through 9, which suggest a further application of the normal form, and Exercises 12 through 17, which furnish practice in transforming equations from representations in one coordinate system to the other.

These last exercises open questions which will be considered in detail in Chapters 5 and 6. In the algebraic manipulation of polar equations we may frequently do some rather wild things which would get us into trouble in rectangular representations. The freedom we exploit stems from three considerations:

i) the multiplicity of the polar representations of a point,
ii) related polar equations, (See Chapter 5.)
iii) "factoring" equations. (See Chapter 6.)

For example, in Exercise 13 we suggest multiplication of both members of the equation by r . In rectangular representations such multiplication by a factor containing a variable is quite likely to add points to the graph, but here the points $(0,\theta)$, which might be added, are already included by the original representation as $(0,(n + \frac{1}{2})\pi)$, where n is any integer.

In Exercise 12 we first obtain

$$r^2 = 36 \text{ , or } r^2 - 36 = (r - 6)(r + 6) = 0 .$$

Now the equations obtained by setting the factors of the left member equal to zero,

$$r = 6 \text{ and } r = -6 ,$$

are related polar equations (as defined on page 167 of the text), for they each have the same graph as $r^2 = 36$. Since each is a simpler representation of the graph, later on we shall prefer either one to the first equation.

In Exercise 17 we first obtain

$$(r^2 + r \sin \theta)^2 = r^2 .$$

If we divide both members by r^2, we obtain

$$(r + \sin \theta)^2 = 1 ,$$

but we have not lost any points from the graph. The pole is the only point we might have lost, and it is still represented by

$$(0,(n + \tfrac{1}{2})\pi) ,$$

where n is any integer. Then we may factor to obtain

$$(r + \sin \theta - 1)(r + \sin \theta + 1) = 0 ;$$

the equations

$$r = 1 - \sin \theta \text{ and } r = -(1 + \sin \theta) ,$$

which are suggested by the factors of the original equation, are related polar equations. Their graphs are identical to the graph of the original equation, and either one is a far simpler representation.

In summary, multiplication or division of both members of an equation by a factor containing the variable and taking the square roots of both members of the equation, are techniques which are fraught with danger and seldom desirable in rectangular representations. They are more frequently acceptable and even desirable in polar representations.

However, we are not suggesting that the teacher should open these questions now. They will be considered in Chapters 5 and 6. To discuss them now would probably only confuse the students. We prefer that the answers to the exercises here be left in the original form obtained without any attempt at simplification. Rather we include this discussion to alert the teacher to the questions laid open and to prepare him or her for the questions that may arise from curious and inquiring students.

Exercises 2-8

1. (a) $-\frac{4}{5}x + \frac{3}{5}y - 3 = 0$

(b) $\frac{5}{13}x + \frac{12}{13}y - 5 = 0$

(c) $\frac{3}{\sqrt{13}}x - \frac{2}{\sqrt{13}}y - \frac{6}{\sqrt{13}} = 0$

(d) $\frac{-5}{\sqrt{34}}x + \frac{3}{\sqrt{34}}y - \frac{12}{\sqrt{34}} = 0$

(e) $\frac{3}{\sqrt{10}}x - \frac{1}{\sqrt{10}}y - \frac{7}{\sqrt{10}} = 0$

(f) $\frac{8}{17}x + \frac{15}{17}y - \frac{30}{17} = 0$

(g) $\frac{12}{13}x - \frac{5}{13}y = 0$

(h) $y - \frac{20}{7} = 0$

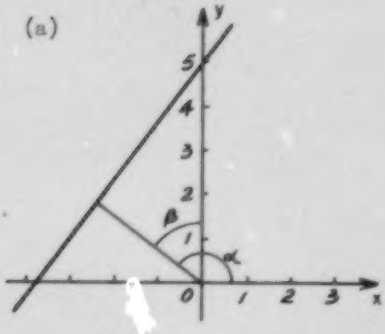
(i) $-x - \frac{15}{9} = 0$

(j) $\frac{5x}{13} - \frac{12y}{13} - \frac{60}{13} = 0$

(k) $-\frac{8}{17}x + \frac{15}{17}y - \frac{120}{17} = 0$

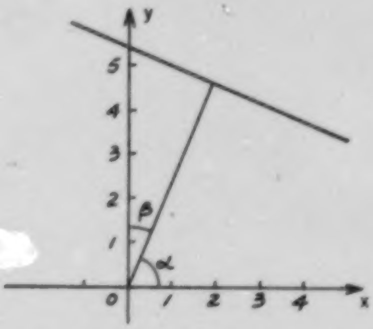
(l) $\frac{3}{5}x - \frac{4}{5}y - \frac{7}{5} = 0$

2. (a)



$4x - 3y + 15 = 0$

(b)



$5x + 12y - 65 = 0$

Of course this is not an efficient way to draw the graph. The exercise was put in to help familiarize the students with this form of equation for a line.

3. (a) $r \sin\theta = 4$

(b) $r \cos\theta = 4$

(c) $\theta = 60^\circ$, or $\theta = \frac{\pi}{3}$

(d) $r\cos(\theta - 315^\circ) = 3$

(e) $r\cos(\theta - 300^\circ) = \frac{3}{2}$

(f) $\theta = 45^\circ$, or $\theta = \frac{\pi}{4}$

(g) $r\cos(\theta - 150^\circ) = 2$

(h) $r\cos(\theta + 135^\circ) = 2$

4. (a) $r\cos\theta - 4 = 0$

(b) $r\sin\theta + 4 = 0$

(c) $\theta = 90^\circ$, or $\theta = \frac{\pi}{2}$

(d) $r\cos\theta + r\sin\theta + 2 = 0$

(e) $3r\cos\theta - 2r\sin\theta + 6 = 0$

(f) $r\cos\theta + \sqrt{3}\, r\sin\theta - 2 = 0$

(g) $15r\sin\theta - 8r\cos\theta + 34 = 0$

5. (a) If P_1 is on L, then $|\lambda x_1 + \mu y_1 - p| = 0$. But the distance from P_1 to L is zero when P_1 is on L.

   (b) P_1 is on the same side of L as O; P_1 is closer than O to L. In this case $d(P_1,F) = p - p_1 = |\lambda x_1 + \mu y_1 - p|$.

   (c) P_1 is on the same side of L as O; P_1 and O are equidistant from L. In this case L_1 contains the origin, $p_1 = 0$, and $d(P_1,F) = p - p_1 = |\lambda x_1 + \mu y_1 - p|$.

6. (a) $\frac{58}{13}$

   (b) $\frac{22}{5}$

   (c) $\frac{20}{\sqrt{17}}$

   (d) $\frac{50}{\sqrt{74}}$

   (e) 0

7. A point $P_0 = (x_0,y_0)$ on the bisector if the distance from P to L_1 is equal to the distance from P to L_2.

   Then from our distance formula, we have

$$\left|\frac{3}{5}x - \frac{4}{5}y + 1\right| = \left|\frac{12}{13}x + \frac{5}{13}y - 1\right|$$

   Taking both choices for the signs yields the two desired equations:

$$21x + 77y - 130 = 0$$

   and

$$11x - 3y = 0$$

8.

$$7x + 9y - 152 = 0$$

   and

$$99x - 77y - 144 = 0$$

9.

$$|\lambda_1 x + \mu_1 y - p_1| = |\lambda_2 x + \mu_2 y - p_2| \quad \text{gives us}$$

$$(\lambda_1 - \lambda_2)x + (\mu_1 - \mu_2)y - (p_1 - p_2) = 0 \quad \text{and}$$

$$(\lambda_1 + \lambda_2)x + (\mu_1 + \mu_2)y - (p_1 + p_2) = 0 .$$

10. $x - 3 = 0$

11. $r \cos \theta - r \sin \theta = 0$

12. $r^2 = 36$

13. $r = 4 \cos \theta$

$r^2 = 4 r \cos \theta$

$(x^2 + y^2) = 4x$

$x^2 - 4x + y^2 = 0$

When $\theta = \frac{\pi}{2}$, $r = 0$. Thus the pole is in the graph of the original equation. One must make this check because both sides of the equation have been multiplied by r; $r = 0$ is then a root of the new equation.

14. $r = 2a \cos \theta$
Note that the pole is in the graph of the equation. Then $r^2 = 2ar \cos \theta$ or $x^2 + y^2 = 2ax$.

15. (a) $y = \sqrt{3}\, x$

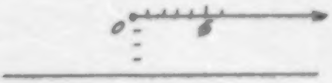
(b) $y + 4 = 0$

(c) $\sqrt{x^2 + y^2} = 5$

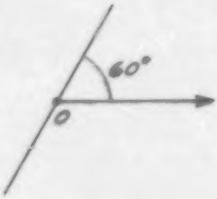
$x^2 + y^2 = 25$

16.

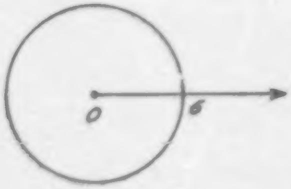
(b)



(a)



(c)



17. (a) $r^2 - 4r \cos \theta = 0$

(b) Note that the pole is in the graph of the equation. Then

$$r^2 = 5r \cos \theta - 3r \sin \theta$$

$$x^2 + y^2 - 5x + 3y = 0$$

(c) $-y = 4$,
or $y = -4$.

(d) $(r^2 + r \sin \theta)^2 = r^2$

Review Exercises - Section 2-6 through Section 2-8

1.

	direction numbers	direction cosines	direction angles (approximately)
(a)	(7,-10)	$(\frac{7}{\sqrt{149}}, -\frac{10}{\sqrt{149}})$	$\alpha = 55^\circ$, $\beta = 145^\circ$
(b)	(25,24)	$(\frac{25}{\sqrt{1201}}, \frac{24}{\sqrt{1201}})$	$\alpha = 44^\circ$, $\beta = 46^\circ$
(c)	(-6,5)	$(\frac{-6}{\sqrt{61}}, \frac{5}{\sqrt{61}})$	$\alpha = 140^\circ$, $\beta = 50^\circ$
(d)	(7,6)	$(\frac{7}{\sqrt{85}}, \frac{6}{\sqrt{85}})$	$\alpha = 41^\circ$, $\beta = 49^\circ$
(e)	(3,-3)	$(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$	$\alpha = 45^\circ$, $\beta = 135^\circ$
(f)	(4,7)	$(\frac{4}{\sqrt{65}}, \frac{7}{\sqrt{65}})$	$\alpha = 60^\circ$, $\beta = 30^\circ$
(g)	(1,2)	$(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}})$	$\alpha = 63^\circ$, $\beta = 27^\circ$
(h)	(-2,1)	$(\frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}})$	$\alpha = 153^\circ$, $\beta = 63^\circ$

2. The points are collinear if two line segments determined by the points have the same slope.

(a) $\frac{13-1}{11-(-4)} = \frac{12}{15} = \frac{4}{5}$

$\frac{13-5}{11-1} = \frac{8}{10} = \frac{4}{5}$ points are collinear

(b) $\frac{-2-7}{1-(-5)} = \frac{-9}{6} = -\frac{3}{2}$

$\frac{-2-(-12)}{1-6} = \frac{10}{-5} = -\frac{2}{1}$ points are not collinear

(c) $\frac{17-(-1)}{23-(-1)} = \frac{18}{24} = \frac{3}{4}$

$\frac{17-(-13)}{23-(-17)} = \frac{30}{40} = \frac{3}{4}$ points are collinear

(d) $\frac{-4-8}{0-(-3)} = \frac{-12}{3} = -4$

$\frac{-4-(-11)}{0-5} = \frac{7}{-5}$ points are not collinear

3. $d(A,B) = \sqrt{41}$ $d(A,C) = \sqrt{53}$ $d(B,C) = 2\sqrt{10}$

4. $\overleftrightarrow{AB} : 4x - 5y + 17 = 0$

$\overleftrightarrow{AC} : 2x + 7y - 1 = 0$

$\overleftrightarrow{BC} : 3x + y - 11 = 0$

5. length of altitude from A : $\frac{19}{\sqrt{10}}$

length of altitude from B : $\frac{38}{\sqrt{53}}$

length of altitude from C : $\frac{38}{\sqrt{41}}$

6. area $(\triangle ABC) = 19$

7. (a) $x(2\sqrt{41} - 4\sqrt{53}) + y(7\sqrt{41} + 5\sqrt{53}) - (\sqrt{41} + 17\sqrt{53}) = 0$

(b) $x(4\sqrt{10} + 3\sqrt{41}) + y(-5\sqrt{10} + \sqrt{41}) + (17\sqrt{10} - 11\sqrt{41}) = 0$

(c) $x(2\sqrt{10} + 3\sqrt{53}) + y(7\sqrt{10} + \sqrt{53}) - (\sqrt{10} + 11\sqrt{53}) = 0$

8. (a) $d(A,L_1) = \frac{3}{\sqrt{13}}$ $\quad d(A,L_2) = \frac{17}{5}$ $\quad d(A,L_3) = \frac{1}{\sqrt{5}}$

(b) $d(B,L_1) = \frac{5}{\sqrt{13}}$ $\quad d(B,L_2) = \frac{14}{5}$ $\quad d(B,L_3) = \frac{4}{\sqrt{5}}$

(c) $d(C,L_1) = \frac{17}{\sqrt{13}}$ $\quad d(C,L_2) = \frac{4}{5}$ $\quad d(C,L_3) = \frac{10}{\sqrt{5}}$

9. (a) $x(10 - 3\sqrt{13}) + y(-15 - 4\sqrt{13}) + (30 + 12\sqrt{13}) = 0$
$x(10 + 3\sqrt{13}) + y(-15 + 4\sqrt{13}) + (30 - 12\sqrt{13}) = 0$

(b) $x(2\sqrt{5} - \sqrt{13}) + y(-3\sqrt{5} + 2\sqrt{13}) + (6\sqrt{5} - 4\sqrt{13}) = 0$
$x(2\sqrt{5} + \sqrt{13}) + y(-3\sqrt{5} - 2\sqrt{13}) + (6\sqrt{5} + 4\sqrt{13}) = 0$

(c) $x(3\sqrt{5} - 5) + y(4\sqrt{5} + 10) + (-12\sqrt{5} - 20) = 0$
$x(3\sqrt{5} + 5) + y(4\sqrt{5} - 10) + (-12\sqrt{5} + 20) = 0$

10. (a) $\frac{6}{\sqrt{13}}$ $\quad$ (b) $\frac{11}{5}$ $\quad$ (c) $\frac{6}{\sqrt{5}}$

11. $P_A(\frac{87}{17}, \frac{92}{17})$ $\quad P_B(-\frac{63}{17}, -\frac{8}{17})$

12. $\theta_1 \quad 82^\circ$
$\theta_2 \quad 98^\circ$

13. L_1 may be written $3x + 5 - 19 = 0$
L_2 may be written $5x - 3y + 7 = 0$

If $a_1a_2 + b_1b_2 = 0$ the lines are perpendicular

Substituting $\quad (3)(5) + (5)(-3) = 0$
and $\quad L_1 \perp L_2$.

14. Find the angles between L_1 and L_2, where L_1 contains the points $(3,4)$, $(-1,-1)$: and L_2 contains the points $(-4,6)$, $(3,0)$.

Solution. Since no sense is imposed on L_1 and L_2 we will find their angles of intersection.

We may take as direction numbers for L_1, $(4,5)$ and for L_2, $(-7,6)$.

(Why?) Therefore:

$$\text{Cos } \theta = \frac{(4)(-7) + (5)(6)}{\sqrt{4^2 + 5^2}\ \sqrt{(-7)^2 + 6^2}} \approx .034$$

$$\therefore \quad \theta \approx 88^{\circ}$$

We may, most simply, find the other angle of intersection as the supplement of θ, but it is instructive to use equivalent direction numbers for L_1 which have the effect of reversing the sense induced by the first choice. We use now $(-4,-5)$, and $(-7,6)$ as pairs of direction numbers and get

$$\text{Cos } \theta' = \frac{(-4)(-7) + (-5)(6)}{\sqrt{(-4)^2 + (-5)^2}\ \sqrt{(-7)^2 + 6^2}} \approx -.034$$

$$\therefore \quad \theta' \approx 92^{\circ}$$

which is, as we expected, supplementary to θ.

15. $$\cos \theta = \frac{\left(-\frac{2}{\sqrt{5}}\right)\left(\frac{7}{\sqrt{58}}\right) + \left(\frac{1}{\sqrt{5}}\right)\left(\frac{3}{\sqrt{58}}\right)}{\sqrt{\frac{4}{5} + \frac{1}{5}}\ \sqrt{\frac{49}{58} + \frac{9}{58}}} \approx -.647$$

$$\theta \approx 130^{\circ}$$

16. $A = (3,4)$ $B = (-2,7)$ $C = (6,9)$

$$m_{\overline{AB}} = \frac{7 - 4}{-2 - 3} = -\frac{3}{5}$$

$$m_{\overline{BC}} = \frac{9 - 7}{6 + 2} = \frac{2}{8} = \frac{1}{4}$$

$$m_{\overline{AC}} = \frac{9 - 4}{6 - 3} = \frac{5}{3}$$

Since $m_{AB}\, m_{AC} = (-\frac{3}{5})(\frac{5}{3}) = -1$

$\overline{AB} \perp \overline{AC}$ and $\triangle ABC$ is a right triangle

17. (a) $-\frac{3}{\sqrt{58}}x + \frac{7}{\sqrt{58}}y - \frac{29}{\sqrt{58}} = 0$

(b) $-\frac{20}{29}x + \frac{21}{29}y - 42 = 0$

(c) $\frac{4}{5}x - \frac{3}{5}y - \frac{24}{5} = 0$

(d) $\frac{3}{\sqrt{58}}x - \frac{7}{\sqrt{58}}y = 0$

(e) $x - \frac{7}{5} = 0$

18. (a) $r \cos (\theta - 60) = 1$
(b) $r \cos \theta = -4$
(c) $\theta = 147^\circ$

19. (a) $\sqrt{3}\, x + y = -5$
(b) $3y - 4x = 12$

20. (a) $r\,(8 \cos \theta + 7 \sin \theta) = 56$
(b) $r\,(15 \sin \theta - 8 \cos \theta) = -180$

Challenge Exercises

1. $3x - 4y + c = 0$ or $ax + by + c = 0$, with $\frac{a}{b} = \frac{3}{-4}$.

2. $4x + 3y + c = 0$ or $ax + by + c = 0$, with $\frac{a}{b} = \frac{4}{3}$.

3. $ax + by = 0$

4. $y - 3 = m(x - 2)$

5. $y = \frac{3}{4}(x - 4)$. (Fixing the value of m reduces the family to one member.)

6. $y = -3x + b$ (a pencil of lines.)

7. Let $L_1 : ax + by + c = 0$ and $L_2 : mx + ny + \rho = 0$ be two intersecting lines. The equations of the lines of the angle bisectors are then

$$x\left(\frac{a}{\sqrt{a^2+b^2}} - \frac{m}{\sqrt{m^2+n^2}}\right) + y\left(\frac{b}{\sqrt{a^2+b^2}} - \frac{n}{\sqrt{m^2+n^2}}\right) + \left(\frac{c}{\sqrt{a^2+b^2}} - \frac{p}{\sqrt{m^2+n^2}}\right) = 0$$

$$x\left(\frac{a}{\sqrt{a^2+b^2}} + \frac{m}{\sqrt{m^2+n^2}}\right) + y\left(\frac{b}{\sqrt{a^2+b^2}} + \frac{n}{\sqrt{m^2+n^2}}\right) + \left(\frac{c}{\sqrt{a^2+b^2}} + \frac{p}{\sqrt{m^2+n^2}}\right) = 0$$

Their slopes are $\dfrac{m\sqrt{a^2 + b^2} - a\sqrt{m^2 + n^2}}{b\sqrt{m^2 + n^2} - n\sqrt{a^2 + b^2}}$, $\dfrac{-m\sqrt{a^2 + b^2} - a\sqrt{m^2 + n^2}}{b\sqrt{m^2 + n^2} + n\sqrt{a^2 + b^2}}$

The product of the slopes is $\dfrac{-m^2(a^2 + b^2) + a^2(m^2 + n^2)}{b^2(m^2 + n^2) - n^2(a^2 \;\; b^2)} = \dfrac{-m^2b^2 + a^2n^2}{m^2b^2 - a^2n^2} = -1$

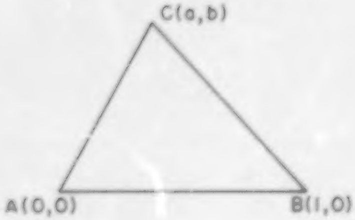
Hence, the lines of the bisectors are perpendicular.

8. $L = \{(x,y): \; ax + by + c = f(x,y) = 0\}$ and

$L_1 = \{(x,y): \; ax_1 + by_1 + c = f(x_1,y_1) = 0\}$

The direction numbers of each line are (a,b) . Therefore the lines are parallel.

9. Given $\triangle ABC$ with vertices $A(0,0)$, $B(1,0)$ and $C(a,b)$.
To prove that the altitudes are congruent at a point H and find the coordinates of H .



the slope of $\overline{AB}$ is 0

the slope of $\overline{AC}$ is $\dfrac{b}{a}$

the slope of $\overline{BC}$ is $\dfrac{b}{a - 1}$

The slope of the altitude from A is $-\dfrac{a - 1}{b}$

The slope of the altitude from B is $-\dfrac{a}{b}$

The altitude from A is represented by $y = -\dfrac{a - 1}{b}x$

The altitude from B is represented by $y = -\dfrac{a}{b}(x - 1)$

If the altitudes are concurrent, $-\dfrac{a - 1}{b}x = -\dfrac{a}{b}(x - 1)$

and $x = a$ and $y = -\dfrac{a(a - 1)}{b}$

the equation of the altitude from C is $x = a$ and the point of intersection of the other two altitudes is clearly on this line.

10. The midpoint of $\overline{AB} = (\frac{1}{2}, 0)$

The midpoint of $\overline{BC} = (\frac{a + 1}{2}, \frac{b}{2})$

The midpoint of $\overline{AC} = (\frac{a}{2}, \frac{b}{2})$

The median from A is represented by

$$y = (\frac{b}{a} + 1)x$$

The median from B is represented by

$$y = \frac{b}{a - 2}(x - 1)$$

These two medians intersect at the point

$$(\frac{a + 1}{3}, \frac{b}{3})$$

The median from C is represented by

$$y = \frac{b}{a - \frac{1}{2}}(x - \frac{1}{2})$$

and the point $(\frac{a + 1}{3}, \frac{b}{3})$ is contained in this line.

Therefore the medians are concurrent at $(\frac{a + 1}{3}, \frac{b}{3})$

11. The bisector of $\angle A$ is given by

$$y = \frac{bx - ay}{\sqrt{a^2 + b^2}}$$ and solving for y,

$$y = \frac{bx}{\sqrt{a^2 + b^2} + a} \qquad (1)$$

The bisector of $\angle B$ is given by

$$y = \frac{b - bx - (1 - a)y}{\sqrt{b^2 + (1 - a)^2}}$$ and solving for y,

$$y = \frac{b(1 - x)}{\sqrt{b^2 + (1 - a)^2} - a + 1} \qquad (2)$$

Equating (1) and (2)

$$\frac{bx}{\sqrt{a^2 + b^2} + a} = \frac{b(1 - x)}{\sqrt{b^2 + (1 - a)^2} + 1 - a}$$

Solving for x we get,

$$x = \frac{\sqrt{a^2 + b^2} + a}{\sqrt{b^2 + (1 - a)^2} + 1 + \sqrt{a^2 + b^2}}$$

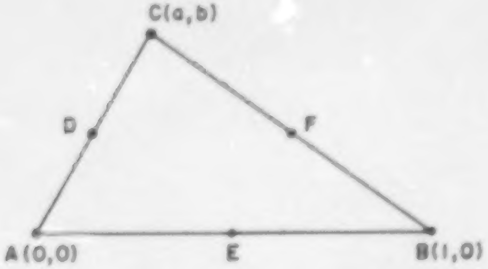
Substituting x into equation (1) ,

$$y = \frac{b}{\sqrt{b^2 + (1 - a)^2} + 1 + \sqrt{a^2 + b^2}}$$

So the point of intersection is

$$\frac{\sqrt{a^2 + b^2} + a}{\sqrt{b^2 + (1 - a)^2} + 1 + \sqrt{a^2 + b^2}} , \frac{b}{\sqrt{b^2 + (1 - a)^2} + 1 + \sqrt{a^2 + b^2}}$$

12.



Midpoint of $\overline{AC} = (\frac{a}{2}, \frac{b}{2}) = D$

Midpoint of $\overline{BC} = (\frac{a + 1}{2}, \frac{b}{2}) = F$

Midpoint of $\overline{AB} = (\frac{1}{2}, 0) = E$

Slope of $\overline{AB} = 0$

Slope of $\overline{AC} = \frac{b}{a}$

Slope of $\overline{BC} = \frac{b}{a - 1}$

Equations of perpendicular bisector through D =

$$y = -\frac{a}{b}x + \frac{a^2}{2b} + \frac{b}{2} \qquad (1)$$

Equation of perpendicular bisector through E =

$$x = \frac{1}{2} \qquad (2)$$

Equation of perpendicular bisector through F =

$$y = -\frac{a-1}{b}x + \frac{a^2-1}{2b} + \frac{b}{2} \qquad (3)$$

If $x = \frac{1}{2}$ is substituted into equation (1) and (3) the values of y are the same. Therefore the perpendicular bisectors are concurrent at

$$\left(\frac{1}{2}, \frac{a^2-a}{2b} + \frac{b}{2}\right)$$

13.

$$H = \left(a, \frac{a(1-a)}{b}\right)$$

$$G = \left(\frac{a+1}{3}, \frac{b}{3}\right)$$

$$E = \left(\frac{1}{2}, \frac{a^2+b^2-a}{2b}\right)$$

The slope of $\overline{HG} = \dfrac{\frac{a-a^2}{b} - \frac{b}{3}}{a - \frac{a+1}{3}} = \dfrac{3a - 3a^2 - b^2}{(2a-1)b}$;

The slope of $\overline{HE} = \dfrac{\frac{a-a^2}{b} - \frac{a^2+b^2-a}{2b}}{a - \frac{1}{2}} = \dfrac{3a - 3a^2 - b^2}{(2a-1)b}$.

Therefore, the points are collinear. An equation of the line is

$$(3a^2 + b^2 - 3a)x + (2ab - b)y + a - a^3 - ab^2 = 0 .$$

Illustrative Test Items - Sections 2-6 through 2-8

1. Find a pair of direction numbers for the line $\overleftrightarrow{PQ}$.

   (a) $P = (2,3)$, $Q = (4,5)$.
   (b) $P = (1,-4)$, $Q = (7,4)$.
   (c) $P = (-2,7)$, $Q = (4,3)$.
   (d) $P = (-2,-3)$, $m = -1$.
   (e) $P = (-1,7)$, $\alpha = 150^\circ$
   (f) x-intercept 4 ; y-intercept 3 .

2. Find a pair of direction cosines for a line,

   (a) $L = \{(x,y): x - y + 2 = 0\}$.
   (b) containing $(3,5)$ and $(1,7)$.
   (c) with slope $-\sqrt{3}$.
   (d) with inclination $\alpha = 30^\circ$.
   (e) parallel to the x-axis .
   (f) perpendicular to the x-axis .

3. Find direction angles for

   (a) the line containing $(-1,-3)$ and $(-3,-1)$.

   (b) the ray emanating from the origin and containing the point $(6,-6\sqrt{3})$.

   (c) the line with equation $\sqrt{3}\,x + y - 7 = 0$.

   (d) the normal segment to $L = \{(x,y): x + \sqrt{3}\,y + 7 = 0\}$.

4. Which, if any, of the lines with the given equations are parallel? perpendicular? the same line?

   L_1: $y - 1 = \frac{2}{3}(x + 2)$
   L_2: $\frac{x}{4} + \frac{y}{6} = 1$
   L_3: $3x + 2y + 3 = 0$
   L_4: $y = \frac{2}{3}x - \frac{1}{3}$
   L_5: $\frac{x + 2}{1 + 2} = \frac{y - \text{[illegible]}}{3 - 1}$

5. Find the cosine of the least angle between the pairs of lines with the indicated equations.

   (a) $x + 3y - 1 = 0$; $2x + 5y - 7 = 0$.
   (b) $2x + 4y - 5 = 0$; $3x + 4y - 1 = 0$.
   (c) $x - y + 13 = 0$; $5x + 3y + 12 = 0$.

6. Let $L = \{(x,y): 4x - 7y + 13 = 0\}$. Write an equation in general form of a line

(a) parallel to L and containing the point $(3,2)$.

(b) perpendicular to L and containing the origin.

(c) parallel to L and with x-intercept 4 .

(d) perpendicular to L and containing the point $(3,2)$.

7. Find an equation of the perpendicular bisector of $\overline{AB}$, where $A = (1,-3)$, $B = (7,1)$.

8. Let $A = (1,1)$, $B = (8,3)$, and $C = (5,8)$. Find the area of triangle ABC .

9. A line L_1 makes an angle whose cosine is $\frac{2}{5}\sqrt{5}$ with $L_2 = \{(x,y): 2x + y - 7 = 0\}$. What is the slope of L_1 ? Find an equation of L_1 if it contains the point $(-4,2)$.

10. Find the normal form of each of the following equations.

(a) $3x - 4y + 15 = 0$

(b) $\dfrac{x - 2}{5 - 2} = \dfrac{y + 1}{2 + 1}$

(c) $y - 7 = \dfrac{7}{3}(x + 4)$

(d) $\dfrac{x}{5} + \dfrac{y}{12} = 1$

(e) $y = \dfrac{8}{15}x - 2$

(f) $\dfrac{x + 3}{21 + 3} = \dfrac{y - 4}{11 - 4}$

(g) $7x - 2y = 0$

(h) $7 - 3y = 0$

11. Find the distance between P and L :

(a) $P = (5,10)$; $L = \{(x,y): 3x - 4y + 10 = 0\}$.

(b) $P = (5,-1)$; $L = \{(x,y): 12x - 5y + 26 = 0\}$.

(c) $P = (6,4)$; $L = \{(x,y): x + 2y - 4 = 0\}$.

(d) $P = (7,-3)$; $L = \{(x,y): 2x - 3y + 5 = 0\}$.

12. Find equations of the lines bisecting the angles formed by

(a) $L_1 = \{(x,y): 3x - 4y + 5 = 0\}$ and $L_2 = \{(x,y): 5x - 12y + 26 = 0\}$

(b) $L_1 = \{(x,y): x + y - 1 = 0\}$ and $L_2 = \{(x,y): 8x - 15y + 34 = 0\}$.

13. Write in polar form the equations of the following lines:

(a) parallel to the polar axis and 2 units above it.

(b) perpendicular to the polar axis and 3 units to the right of the pole.

(c) containing the point $(-2,\frac{5\pi}{4})$ and having inclination $\frac{3\pi}{4}$.

(d) through the pole with slope -1 .

14. Transform each of the following equations into polar coordinates.

(a) $3x - 2y + 5 = 0$

(b) $7x + 8y - 56 = 0$

(c) $x^2 + y^2 = 25$

(d) $y = x^2 + 4x + 4$

15. Transform each of the following equations into rectangular coordinates.

(a) $r \cos \theta = 4$

(b) $2r \cos \theta + 5r \sin \theta = 6$

(c) $r = 3 \sin \theta$

(d) $r \cos (\theta - \frac{\pi}{2}) = 4$

16. Let the vertices of the triangle ABC be $A = (-4,2)$, $B = (6,6)$, $C = (4,-4)$.

(a) Find the lengths of the sides.

(b) Find the equations of the lines containing the sides.

(c) Find an equation of the perpendicular bisector of side $\overline{AC}$.

(d) Find an equation of the line containing the altitude to side $\overline{AC}$.

(e) Find the length of the altitude to side $\overline{AC}$.

(f) Find an equation of the line containing the median to side $\overline{AC}$.

(g) Find the length of the median to side $\overline{AC}$.

(h) Find the area of the triangle.

(i) Find the centroid of triangle ABC (intersection of the medians).

(j) Find an equation of the line containing the bisector of $\angle A$.

Answers

1. (a) $(2,2)$, or equivalent pair
   (b) $(6,8)$, or equivalent pair
   (c) $(6,-4)$, or equivalent pair
   (d) $(1,-1)$, or equivalent pair.
   (e) $(-\sqrt{3},1)$, or equivalent pair.
   (f) $(-4,3)$, or equivalent pair.

2. (a) $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, or $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$
   (b) $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, or $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$
   (c) $(\frac{1}{2}, -\frac{\sqrt{3}}{2})$, or $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$
   (d) $(\frac{\sqrt{3}}{2}, \frac{1}{2})$, or $(-\frac{\sqrt{3}}{2}, -\frac{1}{2})$
   (e) $(1,0)$, or $(-1,0)$
   (f) $(0,1)$, or $(0,-1)$

3. (a) $\alpha = 135^\circ$, $\beta = 45^\circ$; or $\alpha = 45^\circ$, $\beta\ 135^\circ$.
   (b) $\alpha = 60^\circ$, $\beta = 150^\circ$.
   (c) $\alpha = 120^\circ$, $\beta = 30^\circ$; or $\alpha = 60^\circ$, $\beta = 150^\circ$.
   (d) $\alpha = 120^\circ$, $\beta = 150^\circ$.

4. L_1 and L_5 are the same.
   L_1 , L_4 , and L_5 are parallel
   L_2 and L_3 are parallel
   L_1 , L_4 , and L_5 are perpendicular to L_2 and L_3 .

5. (a) $\frac{11}{\sqrt{130}}$ (b) $\frac{11}{5\sqrt{5}}$ (c) $\frac{1}{\sqrt{17}}$

6. (a) $4x - 7y + 2 = 0$
   (b) $7x + 4y = 0$
   (c) $4x - 7y - 16 = 0$
   (d) $7x + 4y - 29 = 0$

7. $3x + 2y - 10 = 0$

8. $20\frac{1}{2}$

9. $m = -\frac{3}{4}$
   $3x + 4y + 4 = 0$

10. (a) $-\frac{3}{5}x + \frac{4}{5}y - 3 = 0$

(b) $\frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}} - \frac{3}{\sqrt{2}} = 0$

(c) $\frac{-7}{\sqrt{58}}x + \frac{3}{\sqrt{58}}y - \frac{49}{\sqrt{58}} = 0$

(d) $\frac{12}{13}x + \frac{5}{13}y - \frac{60}{13} = 0$

(e) $\frac{8}{17}x - \frac{15}{17}y - \frac{30}{17} = 0$

(f) $-\frac{7}{25}x + \frac{24}{25}y - \frac{117}{25} = 0$

(g) $\frac{7}{\sqrt{53}}x - \frac{2}{\sqrt{53}}y = 0$

(h) $y - \frac{7}{3} = 0$

11. (a) 3
(b) 7
(c) $2\sqrt{5}$
(d) $\frac{28}{\sqrt{13}}$

12. (a) $14x + 8y - 65 = 0$ and $64x - 112y + 195 = 0$

(b) $x(17 - 8\sqrt{2}) + y(17 + 15\sqrt{2}) - (17 + 34\sqrt{2}) = 0$ and

$x(17 + 8\sqrt{2}) + y(17 - 15\sqrt{2}) - (17 - 34\sqrt{2}) = 0$.

13. (a) $r \cos (\theta - \frac{\pi}{2}) = 2$

(b) $r \cos \theta = 3$

(c) $r \cos (\theta - \frac{\pi}{4}) = 2$

(d) $\theta = \frac{3\pi}{4}$

14. (a) $3r \cos \theta - 2r \sin \theta + 5 = 0$
(b) $7r \cos \theta + 8r \sin \theta - 56 = 0$

(c) $r^2 = 25$

(d) $r \sin \theta = r^2 \cos^2 \theta + 4r \cos \theta + 4 = (r \cos \theta + 2)^2$

15. (a) $x = 4$

(b) $2x + 5y = 6$

(c) $x^2 + y^2 = 3y$

(d) $y = 4$

16. (a) $d(A,B) = 2\sqrt{29}$; $d(B,C) = 2\sqrt{26}$; $d(A,C) = 10$.

(b) $\overleftrightarrow{AB}$: $2x - 5y + 18 = 0$

$\overleftrightarrow{BC}$: $5x - y - 24 = 0$

$\overleftrightarrow{AC}$: $3x + 4y + 4 = 0$

(c) $4x - 3y - 3 = 0$

(d) $4x - 3y - 6 = 0$

(e) $\frac{46}{5} = 9.2$

(f) $7x - 6y - 6 = 0$

(g) $\sqrt{85}$

(h) 46

(i) $(2,\frac{4}{3})$

(j) $x(3\sqrt{29} - 10) + y(4\sqrt{29} + 25) + (4\sqrt{29} - 90) = 0$

Chapter 3

VECTORS AND THEIR APPLICATION

3-1. Why Study "Vectors"?

91 In the opening paragraphs reference is made to the increasing importance of vectors and vector methods in the fields of applied mathematics, science, and engineering. You need only pick up any text in these subjects to be assured of the accuracy of this statement. Most recent books in calculus (e.g., *Calculus and Analytic Geometry* by G.B. Thomas) make considerable use of vector methods. You may like to read *Analytic Geometry: A Vector Approach* by Charles Wexler for an extensive treatment of this subject.

It is quite likely that most of your students will go on to study calculus and more advanced mathematics. Most students in science and engineering are now encouraged to take courses in vector analysis and linear algebra. The latter course starts with vector algebra and uses it to approach the subject of matrices. In this context, a vector is a row or column of a matrix. Our approach is from the geometric point of view (as is vector analysis) but the two are clearly closely related.

The beginnings of this subject can be found in the writings of Aristotle, and later in the works of Galileo (1564-1642, Italian). However, serious study of the subject began with William Rowan Hamilton (1805-1865, Irish) and Herman Grassmann (1809-1877, German). Their work was dependent upon the earlier development of analytic geometry. Hamilton was inspired by problems arising from Newtonian physics and astronomy. In solving problems related to the motion of particles, Hamilton needed a non-commutative algebra. The quaternion $A = a_0 + a_1 i + a_2 j + a_3 k$ (where $i^2 = j^2 = k^2 = ijk = -1$ and the a's are real), provided the answer since, for example, $i \cdot j = -j \cdot i$. The quaternion led to the vector and, in the cross-product of vectors, $A \times B = -B \times A$. (See this Commentary on Section 3-7).

Grassmann approached the subject of vectors from the algebraic point of view. He was seeking an algebraic method of extending geometry from three into n dimensions. A vector in two dimensions is defined as an ordered pair of real numbers and in three dimensions as an ordered triple of real numbers. In n dimensions, a vector is an ordered n-tuple of real numbers. This is the approach used today in the study of vector spaces in modern algebra.

If your students have already studied vectors in SMSG "Geometry with Coordinates", "Intermediate Mathematics", or "Matrix Algebra", a large part of the material in this chapter will serve as a review. Some time should be spent, however, in analyzing the different approaches to the subject. In this way the students will review the topic from another point of view. Some of the subject matter and many of the problems are new to all.

3-2. Directed Line Segments and Vectors.

92 For more information regarding directed line segments, you should read the SMSG "Intermediate Mathematics", p. 629-634.

Probably the most distinctive part of our approach to the study of vectors lies in our definition of a vector. Since there is no way to distinguish any directed line segment from another with the same magnitude and sense of direction, it is therefore reasonable to define a vector as an infinite set of equivalent directed line segments. Any member of the set can be used to represent this vector. The origin-vector (a new term created here) is very often used to represent the set because of its convenience in geometric proofs and in the study of vector components.

Unless specific geometric conditions obtain, our approach to the subject also gives us the freedom to use free vectors or bound vectors as we choose. The "Origin Principle" on page 93 and the "Origin-Vector Principle" on page 96 are carefully and explicitly stated to make this point clear.

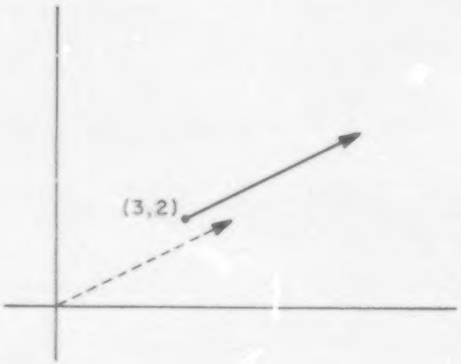
93 The question of equality or inequality of vectors refers only to sets. When we say "two vectors are equal" we are only talking about the same infinite set of directed line segments. Thus "equality" really means "identity". The use of the term in this sense is consistent with its use in all other SMSG texts. For example in earlier texts, if $\overline{AB} = \overline{CD}$, then $\overline{AB}$ and $\overline{CD}$ are identically the same segment, with $A = C$ and $B = D$.

However, in applications of vectors, it is convenient to use the term vector, as we state in the text, to mean a single member of the set. We consider it proper to do this when there is no danger of ambiguity. The students will then be on more familiar ground when they meet vectors in other courses.

96 The discussion surrounding the origin-vector principle is of greatest importance. You will have many occasions to refer to it in the succeeding sections, particularly in Chapter 4, where many proofs of geometric theorems are discussed.

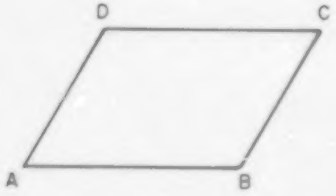
Exercises 3-2

1.



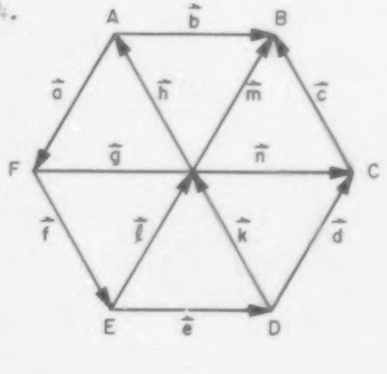
2. $\overrightarrow{FE}$ and $\overrightarrow{JI}$; $\overrightarrow{LK}$ and $\overrightarrow{UT}$; $\overrightarrow{QR}$, $\overrightarrow{OP}$ and $\overrightarrow{MN}$, $\overrightarrow{QS}$ and $\overrightarrow{TV}$
Each set is a representation of the same vector.

3.



$\overrightarrow{DC}$ and $\overrightarrow{AB}$
$\overrightarrow{CD}$ and $\overrightarrow{BA}$
$\overrightarrow{AD}$ and $\overrightarrow{BC}$
$\overrightarrow{DA}$ and $\overrightarrow{CB}$

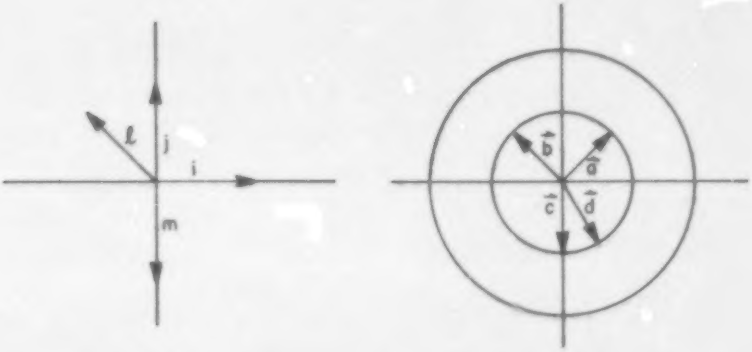
4.



(a) $\vec{\ell} = \vec{m}$
$\vec{e} = \vec{b}$
$\vec{\ell} = \vec{d}$
$\vec{h} = \vec{k}$
(and others)

(b) $\vec{a} = -\vec{\ell}$
$\vec{f} = -\vec{c}$
$\vec{g} = -\vec{n}$
$\vec{h} = -\vec{f}$
(and others)

5.



6. Motion of a car, winds, weight, momentum, angular momentum, electrical and magnetic fields, etc.

3-3. Sum and Difference of Vectors. Scalar Multiplication.

97 The definition presented on this page is concerned only with the sum of two non-zero vectors not lying in the same line.

If $\vec{A}$ and $\vec{B}$ lie in the same line and have the same sense of direction, then $\vec{A} + \vec{B}$ is a vector in the same line with the same sense of direction and with magnitude $|\vec{A}| + |\vec{B}|$. If $\vec{A}$ and $\vec{B}$ have different senses of direction and, let us say, $|\vec{A}| > |\vec{B}|$, then $\vec{A} + \vec{B}$ will have the direction of $\vec{A}$ and magnitude $|\vec{A}| - |\vec{B}|$.

98 By part (2) of the definition of the sum of two vectors, $\vec{P} + \vec{P}$ is a vector with magnitude twice the magnitude of $\vec{P}$. Similarly $(\vec{P} + \vec{P}) + \vec{P}$ is a vector with magnitude 3 times the magnitude of $\vec{P}$. Thus the definition of $r\vec{P}$ generalizes naturally from what we think $2\vec{P}$ and $3\vec{P}$ should be (neither being defined at this point).

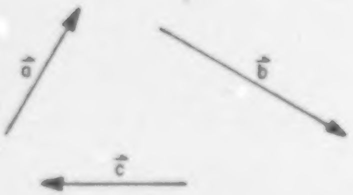
99 An emphasis on subtraction of vectors defined in terms of addition should be made. This should be done not only for purely algebraic reasons, but also to simplify finding the difference of two vectors in a vector diagram.

Exercises 3-3

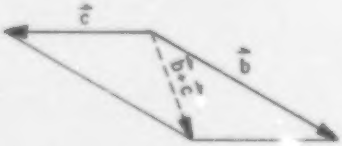
1. (a) $\overrightarrow{C}$
   (b) $\overrightarrow{C}$
   (c) $\overrightarrow{E}$ (requires assumptions that vector addition is associative and that diagonals of a parallelogram bisect each other)
   (d) 2 (requires second assumption in part c)
   (e) $\overrightarrow{C}$

2. (a) $\overrightarrow{c} = -\overrightarrow{a}$
   $\overrightarrow{d} = -\overrightarrow{b}$
   $\overrightarrow{e} = \overrightarrow{a} + \overrightarrow{b}$
   (b) (i) $\overrightarrow{e} = \overrightarrow{a} - \overrightarrow{d}$
   (ii) $\overrightarrow{e} = \overrightarrow{a} + \overrightarrow{b}$
   (iii) $\overrightarrow{e} = \overrightarrow{b} - \overrightarrow{c}$
   (iv) $\overrightarrow{e} = -\overrightarrow{c} - \overrightarrow{d}$
   (c) (i) $\overrightarrow{0}$
   (ii) $\overrightarrow{0}$

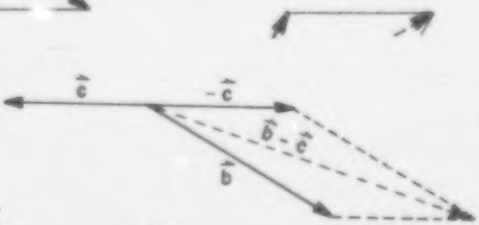
3.


$\overrightarrow{a}$
$\overrightarrow{b}$
$\overrightarrow{c}$

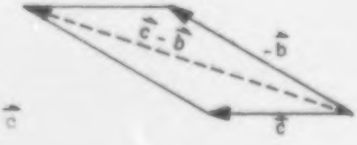
(a) $\overrightarrow{b} + \overrightarrow{c}$


$\overrightarrow{c}$
$\overrightarrow{b}+\overrightarrow{c}$
$\overrightarrow{b}$

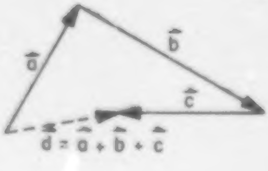
(b) $\overrightarrow{b} - \overrightarrow{c}$


$\overrightarrow{c}$
$-\overrightarrow{c}$
$\overrightarrow{b}-\overrightarrow{c}$
$\overrightarrow{b}$

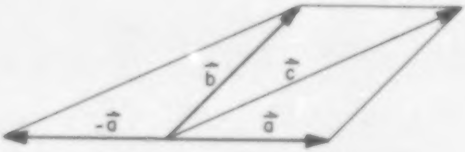
(c) $\overrightarrow{c} - \overrightarrow{b}$


$\overrightarrow{c}-\overrightarrow{b}$
$-\overrightarrow{b}$
$\overrightarrow{c}$

(d) $\overrightarrow{a} + \overrightarrow{b} + \overrightarrow{c}$


$\overrightarrow{a}$
$\overrightarrow{b}$
$\overrightarrow{c}$
$\overrightarrow{d} = \overrightarrow{a} + \overrightarrow{b} + \overrightarrow{c}$

4.


$\overrightarrow{b}$
$\overrightarrow{c}$
$-\overrightarrow{a}$
$\overrightarrow{a}$

$\overrightarrow{a} + \overrightarrow{b} = \overrightarrow{c}$

It can also be seen that $-\overrightarrow{a} + \overrightarrow{c} = \overrightarrow{b}$ $\therefore \overrightarrow{b} = \overrightarrow{c} - \overrightarrow{a}$

5. (a) $\frac{1}{2}$

(b) 2

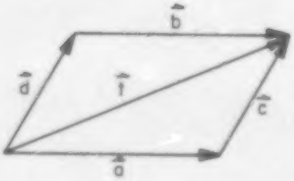
(c) -1

(d) $\frac{3}{4}$

(e) $\frac{3}{2}$

(f) $-\frac{3}{5}$

6.


$\overrightarrow{b}$
$\overrightarrow{d}$
$\overrightarrow{t}$
$\overrightarrow{c}$
$\overrightarrow{a}$

From the diagram above $\overrightarrow{a} = \overrightarrow{b}$ and $\overrightarrow{c} = \overrightarrow{d}$

also $\overrightarrow{b} + \overrightarrow{d} = \overrightarrow{t}$ and $\overrightarrow{a} + \overrightarrow{c} = \overrightarrow{t}$.

$$\therefore \overrightarrow{a} + \overrightarrow{c} = \overrightarrow{b} + \overrightarrow{d}$$

7. $|4\overrightarrow{A}| = 12$

$|-5\overrightarrow{A}| = 15$

$-|5\overrightarrow{A}| = -15$

8. Since $\overrightarrow{a} = \overrightarrow{b}$, $\overrightarrow{a}$ and $\overrightarrow{b}$ are representatives of the same infinite set of equivalent directed line segments. Thus

$$|\overrightarrow{a}| = |\overrightarrow{b}| \text{ and } \overrightarrow{a} \| \overrightarrow{b}$$

Now $r\overrightarrow{a} \| \overrightarrow{a}$ and $r\overrightarrow{a}$ is r times as large as $\overrightarrow{a}$. Also $r\overrightarrow{b} \| \overrightarrow{b}$ and is r times as large as $\overrightarrow{b}$. Thus

$$|r\overrightarrow{a}| = |r\overrightarrow{b}| \text{ and } r\overrightarrow{a} \| r\overrightarrow{b}$$

$$\therefore r\overrightarrow{a} = r\overrightarrow{b}$$

9. $|k\overrightarrow{b}|$ is equal to the magnitude of $\overrightarrow{a}$.

10. (a) $\overrightarrow{b} = \overrightarrow{e}$

$\overrightarrow{h} = \overrightarrow{p}$

$\overrightarrow{g} = -\overrightarrow{n}$

$\overrightarrow{d} = -\overrightarrow{a}$

$\overrightarrow{c} = -\overrightarrow{f}$

$\overrightarrow{\ell} = \overrightarrow{m}$

(b) $\overrightarrow{\ell} - \overrightarrow{k} = \overrightarrow{b}$

$\overrightarrow{g} + \overrightarrow{b} = \overrightarrow{a} + \overrightarrow{m}$

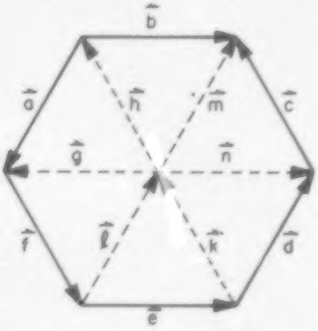
$\overrightarrow{\ell} - \overrightarrow{c} + \overrightarrow{g} = \overrightarrow{0}$

$\overrightarrow{h} + \overrightarrow{n} = \overrightarrow{\ell} + \overrightarrow{k}$

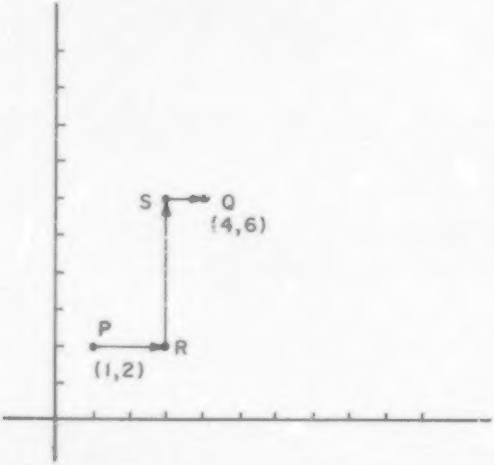
$\overrightarrow{b} + \overrightarrow{e} = 2\overrightarrow{n}$

$\overrightarrow{e} + \overrightarrow{h} = -\overrightarrow{a}$

(and others)


$\overrightarrow{b}$
$\overrightarrow{a}$
$\overrightarrow{h}$
$\overrightarrow{m}$
$\overrightarrow{c}$
$\overrightarrow{g}$
$\overrightarrow{n}$
$\overrightarrow{\ell}$
$\overrightarrow{k}$
$\overrightarrow{f}$
$\overrightarrow{d}$
$\overrightarrow{e}$

11\.

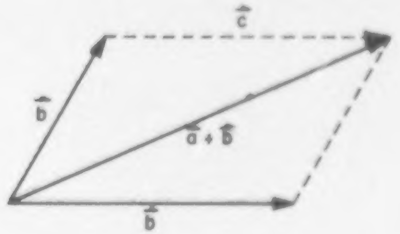


One example: One could follow the path from P to R , from R to S , from S to Q .

12. (a) not necessarily

(b) yes

13.

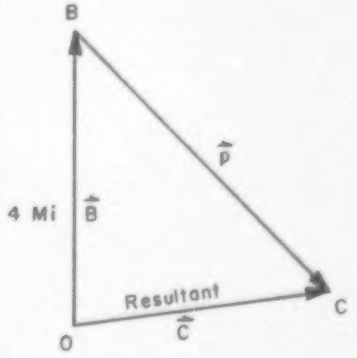

$\vec{c}$
$\vec{b}$
$\vec{a} + \vec{b}$
$\vec{b}$

$|\vec{a}|$ is length of $\vec{a}$
$|\vec{b}|$ is length of $\vec{b}$
$|\vec{a} + \vec{b}|$ is length of $\vec{a} + \vec{b}$

Since $\vec{c}$ is equivalent to $\vec{a}$, then $|\vec{c}| = |\vec{a}|$.
Since the sum of the lengths of two sides of a triangle is greater than or equal that of the third, we have

$$|\vec{c}| + |\vec{b}| \geq |\vec{a} + \vec{b}| .$$
$$\therefore |\vec{a}| + |\vec{b}| \geq |\vec{a} + \vec{b}| .$$

14.


B
$\vec{P}$
4 Mi
$\vec{B}$
Resultant
$\vec{C}$
O
C

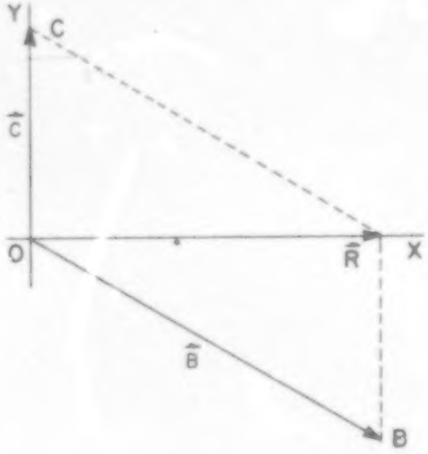
$|\vec{B}| = 2''$

$|\vec{P}| = 2\frac{1}{2}''$

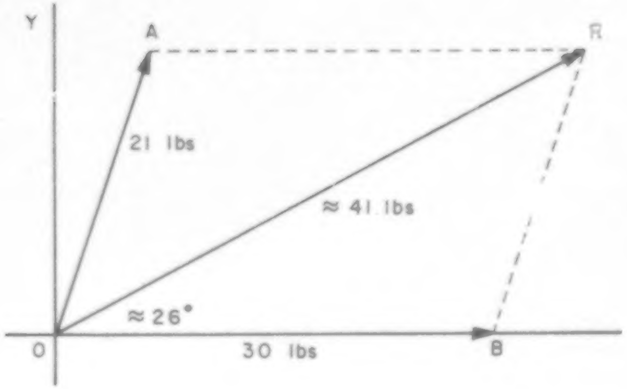
$\vec{C}$ is the resultant

$|\vec{C}| \approx 1\frac{3}{4}''$, representing approximately $3\frac{1}{2}$ miles in the direction indicated.

15. Let the speed and direction of the current be represented by $\vec{C}$ along the y-axis. Let the actual speed and direction of the boat be represented by $\vec{R}$. We want to find the vector $\vec{B}$ representing the boat's motion in still water which when added to $\vec{C}$ represents the combined effect of current and engine on the boat. $\vec{R} = \vec{C} + \vec{B}$. $|\vec{B}|$ represents 6 m.p.h. at $\angle ROB$.



16.



17. $\vec{A}$ and $\vec{B}$ are distinct vectors
Let A have coordinates (a,b), B coordinates (c,d)
Then $-\vec{B}$ has its terminal point at $(-c,-d)$
and $-\vec{A}$ has its terminal point at $(-a,-b)$.
Thus $\vec{A} - \vec{B}$ has its terminal point at $(a - c, b - d)$
and $\vec{B} - \vec{A}$ has its terminal point at $(c - a, d - b)$.

Case one: $b \neq d$

Then slope of line $\overleftrightarrow{AB}$ is given by $\frac{b - d}{a - c}$

and slope of line $\overleftrightarrow{OC}$ is given by $\frac{(b - d) - 0}{(a - c) - 0} = \frac{b - d}{a - c}$.

Therefore the lines are parallel.

Case two: $b = d$.

Then line $\overleftrightarrow{AB}$ has no slope defined, but it is parallel to the line $x = 0$, which is the line OC.

The proof that $\vec{B} - \vec{A}$ lies on a line parallel to the line through A and B is similar.

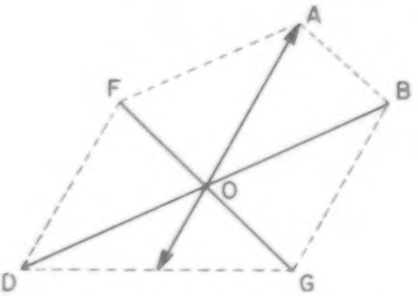
If $b \neq d$ then $m(\overleftrightarrow{AB}) = \dfrac{d - b}{c - a}$

and $m(\overleftrightarrow{OD}) = \dfrac{(d - b) - 0}{(c - a) - 0} = \dfrac{d - b}{c - a}$

So the lines are parallel.

If $b = d$, then $\overleftrightarrow{AB}$ is parallel to the line $x = 0$ which is $\overleftrightarrow{OD}$.

Alternatively, we need not use coordinates:



Let $\vec{D} = -\vec{B}$ and $\vec{E} = -\vec{A}$. $\vec{A} - \vec{B}$ is the vector determined by the vector opposite 0 in the parallelogram formed with $\overline{OA}$ and $\overline{OD}$ as sides. Hence $\vec{F} = \vec{A} - \vec{B}$. But $d(F,A) = d(D,O)$ and $d(D,O) = d(O,B)$. So $d(F,A) = d(O,B)$. Because $\overleftrightarrow{OD} = \overleftrightarrow{OB}$ and $\overleftrightarrow{FA} \parallel \overleftrightarrow{OD}$, we see that $\angle FAO = \angle BOA$. With $d(O,..) = d(A,O)$ we now know that $\triangle FAO \cong \triangle BOA$. We get $d(F,O) = d(a,B)$ which tells us that OFAB is a parallelogram since we already have $d(F,A) = d(O,B)$. So $\vec{F} = \vec{A} - \vec{B}$ lies on a line parallel to $\overleftrightarrow{AB}$.

18. Given that $\vec{a}$, $\vec{b}$, $\vec{c}$, and $\vec{d}$, are consecutive vector sides of a quadrilateral. We wish to prove that the figure is a parallelogram if and only if $\vec{b} + \vec{d} = \vec{0}$. We must show that:

(1) if $\vec{b} + \vec{d} = \vec{0}$, then the quadrilateral is a parallelogram and that
(2) if the quadrilateral is a parallelogram, then $\vec{b} + \vec{d} = \vec{0}$.

Proof:

(1) Assume $\vec{b} + \vec{d} = \vec{0}$

$\vec{b} = -\vec{d}$

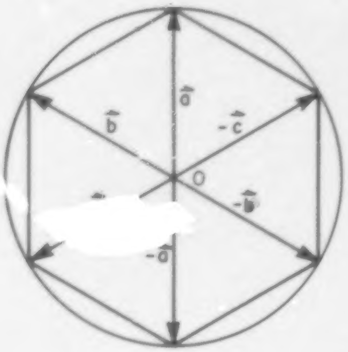
$\therefore$ $\vec{b}$ and $\vec{d}$ are parallel, have the same magnitude and are opposite sides.

$\therefore$ Quadrilateral is a parallelogram.

(2) Assume the quadrilateral is a parallelogram. Then the opposite sides must be equal and parallel; i.e., $\vec{b} = -\vec{d}$.

$\therefore$ $\vec{b} + \vec{d} = \vec{0}$.

19.

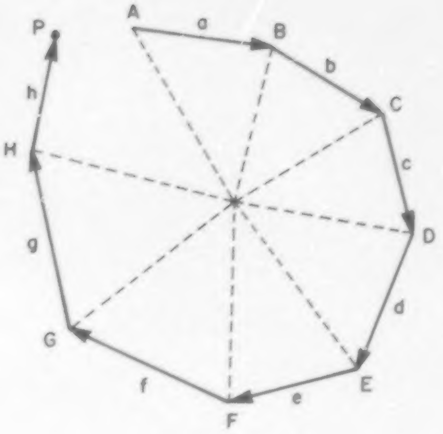


The diagram above shows labeling which leads to a simple proof.

To prove: The sum of six vectors drawn from the center of a regular hexagon to its vertices is zero.

$$\overrightarrow{a} + (-\overrightarrow{a}) + \overrightarrow{b} + (-\overrightarrow{b}) + \overrightarrow{c} + (-\overrightarrow{c}) = \overrightarrow{0} .$$

20.



(1) Let $\overrightarrow{AB} = \overrightarrow{a}$, $\overrightarrow{BC} = \overrightarrow{b}$, $\overrightarrow{CD} = \overrightarrow{c}$, ... , [illegible] $= \overrightarrow{p}$.

(2) Note that for triangle ABO , we have

$$\overrightarrow{AB} + \overrightarrow{BO} = - \overrightarrow{OA}$$
$$\therefore \quad \overrightarrow{AB} + \overrightarrow{BO} + \overrightarrow{OA} = \overrightarrow{0}$$

(3) Then if we divide our polygon into triangles as shown, we have:

$$(\overrightarrow{AB} + \overrightarrow{BO} + \overrightarrow{OA}) + (\overrightarrow{BC} + \overrightarrow{CO} + \overrightarrow{OB}) + \dots + \overrightarrow{AO} = \overrightarrow{0}.$$

But $\overrightarrow{AO} = -\overrightarrow{OA}$, $\overrightarrow{BD} = -\overrightarrow{DB}$, etc. ...

$\therefore$ (4) $\overrightarrow{a} + \overrightarrow{b} + \overrightarrow{c} + \dots + \overrightarrow{p} = \overrightarrow{0}$, or $\overrightarrow{AB} + \overrightarrow{BC} + \dots \overrightarrow{PA} = \overrightarrow{0}$.

3-4. Properties of Vector Operations.

104 The purpose of this section is to develop some algebraic structure for the operations of vector addition and scaler multiplication.

Perhaps the best way of showing the associative property by means of Figure 3-9 is to consider the quadrilateral whose vertices are the terminal points of $\overrightarrow{Q}, \overrightarrow{P} + \overrightarrow{Q}, \overrightarrow{Q} + \overrightarrow{R}$, and $(\overrightarrow{P} + \overrightarrow{Q}) + \overrightarrow{R}$. It is a parallelogram since each of a pair of opposite sides is parallel to $\overrightarrow{R}$ and has length equal to the length of $\overrightarrow{R}$. Similarly the terminal points of $\overrightarrow{R}$, $\overrightarrow{P} + \overrightarrow{Q}$, $\overrightarrow{Q} + \overrightarrow{R}$, and $\overrightarrow{P} + (\overrightarrow{Q} + \overrightarrow{R})$ are vertices of a parallelogram (opposite sides equal in length and parallel to $\overrightarrow{P}$). Thus the two parallelograms are identical and the fourth vertices must coincide.

105 A nicer proof depends on the one-to-one correspondence between points in the plane and ordered pairs of real numbers. It appears in the solution in Exercise 17, Section 3-6.

THEOREM 3-4. The vectors $(rs)\overline{P}$ and $r(s\overline{P})$ both have terminal point X such that $d(0,X) = rs\, d(0,P)$.

Exercises 3-4

1. (a) Show that: $\overrightarrow{B} + (\overrightarrow{A} - \overrightarrow{B}) = \overrightarrow{A}$

If $\overrightarrow{B} + (\overrightarrow{A} - \overrightarrow{B}) = \overrightarrow{A}$,

then $\overrightarrow{B} + (-\overrightarrow{B} + \overrightarrow{A}) = \overrightarrow{A}$

$\left(\overrightarrow{B} + (-\overrightarrow{B})\right) + \overrightarrow{A} = \overrightarrow{A}$

and $\overrightarrow{A} = \overrightarrow{A}$.

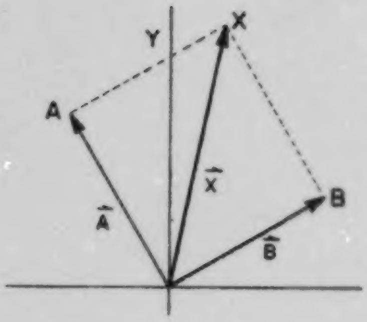
Since this last statement is true, the steps can be reversed to prove that $\overrightarrow{B} + (\overrightarrow{A} - \overrightarrow{B}) = \overrightarrow{A}$.

(b) If $(\overrightarrow{A} - \overrightarrow{B}) + \overrightarrow{B} = \overrightarrow{A}$,

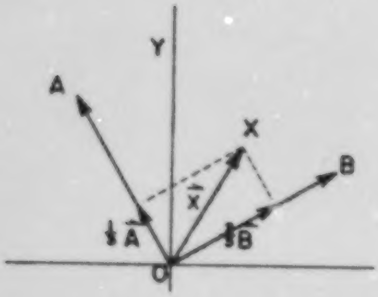
then $\overrightarrow{A} + \left((-\overrightarrow{B}) + \overrightarrow{B}\right) = \overrightarrow{A}$

and $\overrightarrow{A} = \overrightarrow{A}$. (See remark in part (a))

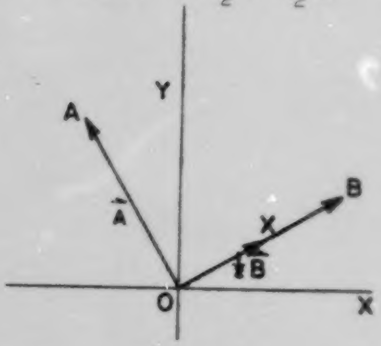
2. (a) $\overleftarrow{X} = 1 \cdot \overleftarrow{A} + 1 \cdot \overleftarrow{B}$


Y
X
A
$\overleftarrow{X}$
B
$\overleftarrow{A}$
$\overleftarrow{B}$

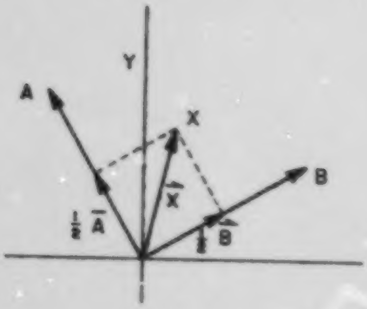
(b) $\overleftarrow{X} = \frac{1}{3}\overleftarrow{A} + \frac{2}{3}\overleftarrow{B}$


Y
A
X
B
$\overleftarrow{X}$
$\frac{1}{3}\overleftarrow{A}$
$\frac{2}{3}\overleftarrow{B}$
O

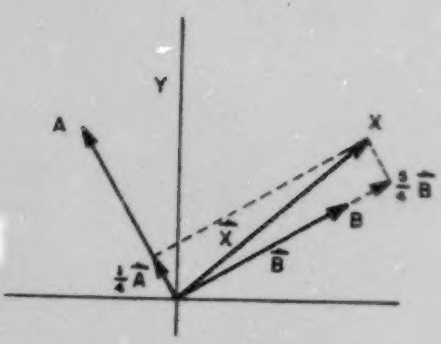
(c) $\overleftarrow{X} = 0 \cdot \overleftarrow{A} + \frac{1}{2}\overleftarrow{B} = \frac{1}{2}\overleftarrow{B}$


Y
A
$\overleftarrow{A}$
B
X
$\frac{1}{2}\overleftarrow{B}$
O
X

(d) $\overleftarrow{X} = \frac{1}{2}\overleftarrow{A} + \frac{1}{2}\overleftarrow{B}$

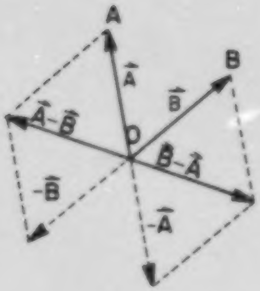

Y
A
X
B
$\overleftarrow{X}$
$\frac{1}{2}\overleftarrow{A}$
$\frac{1}{2}\overleftarrow{B}$

(e) $\overleftarrow{X} = \frac{1}{4}\overleftarrow{A} + \frac{5}{4}\overleftarrow{B}$


Y
A
X
$\frac{5}{4}\overleftarrow{B}$
B
$\overleftarrow{X}$
$\overleftarrow{B}$
$\frac{1}{4}\overleftarrow{A}$

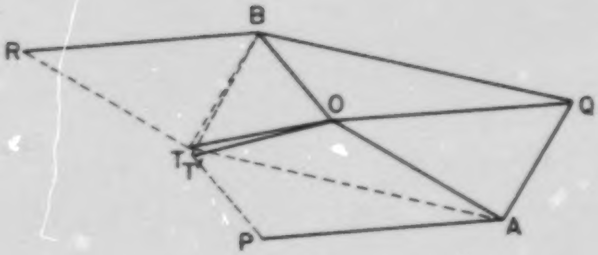
X is on $\overleftrightarrow{AB}$ when the sum of p + q is 1.

3. (a)



(b) $\vec{A} - \vec{B} = r(\vec{B} - \vec{A})$ for $r = -1$

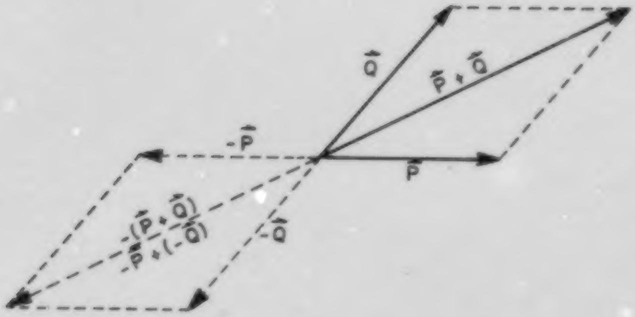
4.



Let O be the origin and points P, Q, R determine vectors $\vec{P}$, $\vec{Q}$ and $\vec{R}$.
Let A be the vertex opposite O in the parallelogram determined by $\vec{R}$ and $\vec{Q}$, i.e., $\vec{A} = \vec{P} + \vec{Q}$.
Let B be the vertex opposite O in the parallelogram determined by $\vec{Q}$ and $\vec{R}$, i.e., $\vec{B} = \vec{Q} + \vec{R}$.
Let $\vec{T} = \vec{A} + \vec{R}$ and $\vec{T} = \vec{P} + \vec{B}$
$= (\vec{P} + \vec{Q}) + \vec{R}$ $= \vec{P} + (\vec{Q} + \vec{R})$
We wish to prove $\vec{T} = \vec{T'}$. It is enough to show that T and T' coincide.
By using Exercises 3-3, Problem 17, $\overline{AT} \parallel \overline{OR} \parallel \overline{QB}$ and
$d(A,T) = d(O,B) = d(Q,B)$.
Thus ATBQ is a parallelogram so $\overline{BT} \parallel \overline{QA}$ and $d(B,T) = d(B,T)$.
By construction of A, $\overline{OP} \parallel \overline{QA}$ and $d(OP) = d(QA)$.
By construction of T', $\overline{BT'} \parallel \overline{OP}$ and $d(B,T') = d(O,P)$.
Therefore $\overline{BT} \parallel \overline{BT'}$ and $d(B,T) = d(B,T')$.
So we must have $\overline{BT} = \overline{BT'}$.
Whence $T = T'$ and $\vec{T} = \vec{T'}$. Q.E.D.

5. $-(\overrightarrow{P} + \overrightarrow{Q}) = -\overrightarrow{P} - \overrightarrow{Q}$

$\qquad = -\overrightarrow{P} + (-\overrightarrow{Q})$



6. If $\qquad (-r)\overrightarrow{P} = r(-\overrightarrow{P})$,

then $\qquad (-r)\overrightarrow{P} = r[(-1)(\overrightarrow{P})]$

$\qquad (-r)\overrightarrow{P} = (r)(-1)(\overrightarrow{P})$

and $\qquad (-r)\overrightarrow{P} = (-r)\overrightarrow{P}$.

Since this last statement is true, the steps can be reversed to prove that $(-r)\overrightarrow{P} = r(-\overrightarrow{P})$.

3-5. Characterization of the Point on a Line.

109 In the proof of the distributive laws (Theorem 3-6), we left two items as unfinished business. The first was the proof in the case where $\overrightarrow{P}$ and $\overrightarrow{Q}$ are collinear and have opposite senses of direction.

In this case, assume $|\overrightarrow{P}| > |\overrightarrow{Q}|$. Then:

(1) By the same definition we used earlier, $\overrightarrow{P} + \overrightarrow{Q}$ has the same direction as $\overrightarrow{P}$ and has magnitude $|\overrightarrow{P}| - |\overrightarrow{Q}|$.

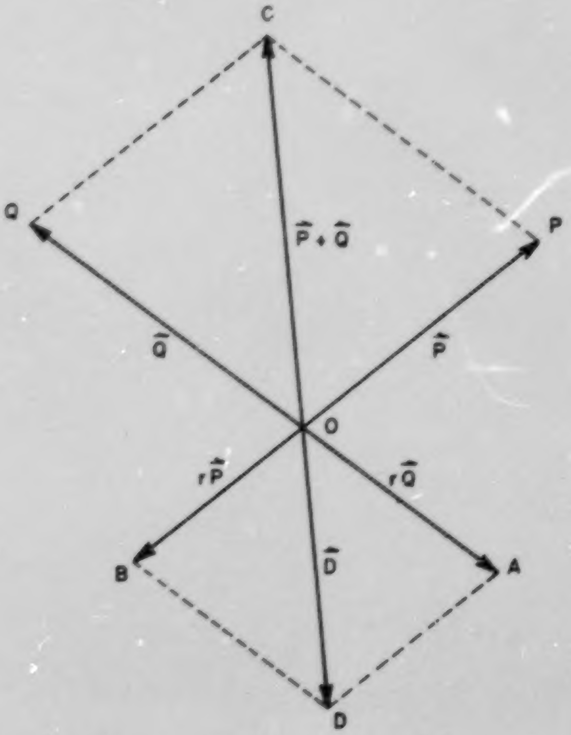
(2) If $r > 0$, then $r(\overrightarrow{P} + \overrightarrow{Q})$ has the same direction as $(\overrightarrow{P} + \overrightarrow{Q})$, and, by (1) above, the same direction as $\overrightarrow{P}$. The magnitude of $r(\overrightarrow{P} + \overrightarrow{Q}) = |r(\overrightarrow{P} + \overrightarrow{Q})| = r|\overrightarrow{P} + \overrightarrow{Q}|$ and is, by (1) above, equal to $r(|\overrightarrow{P}| - |\overrightarrow{Q}|)$. The distributive law gives the magnitude as $r|\overrightarrow{P}| - r|\overrightarrow{Q}|$.

(3) We now consider $r\overrightarrow{P}$ and $r\overrightarrow{Q}$, which, since $r > 0$, have the same directions respectively as $\overrightarrow{P}$ and $\overrightarrow{Q}$. By our hypothesis, $\overrightarrow{P}$ and $\overrightarrow{Q}$ have opposite senses of directions, and therefore so do $r\overrightarrow{P}$ and $r\overrightarrow{Q}$. Since we have assumed $|\overrightarrow{P}| > |\overrightarrow{Q}|$, we have $r|\overrightarrow{P}| > r|\overrightarrow{Q}|$, and, therefore $|r\overrightarrow{P}| > |r\overrightarrow{Q}|$.

(4) Our definition for the sum of vectors now requires that $r\vec{P} + r\vec{Q}$ have the same direction as $r\vec{P}$ and this is the same direction as $\vec{P}$. The same definition requires that the magnitude of $r\vec{P} + r\vec{Q}$ be $|r\vec{P}| - |r\vec{Q}|$; but this latter expression can be written as $r(|\vec{P}| - |\vec{Q}|)$.

(5) Since we have shown that the vectors $r(\vec{P} + \vec{Q})$ and $r\vec{P} + r\vec{Q}$ have the same magnitude and the same sense of direction, we have shown that they are equal.

The second item we did not discuss concerned the proof when $r < 0$. In this case, our figure must be changed to the following:



$\vec{A} = r\vec{Q}$
$\vec{B} = r\vec{P}$

Since $r < 0$, $r\vec{P}$ and $r\vec{Q}$ have directions opposite those of $\vec{P}$ and $\vec{Q}$ respectively. The proof for the case $r > 0$ in the text will need to be modified as follows in order to hold when $r < 0$.

In step (1) , since r is negative and the absolute values positive, $|\vec{A}| = -r|\vec{Q}|$ and $|\vec{B}| = -r|\vec{P}|$.

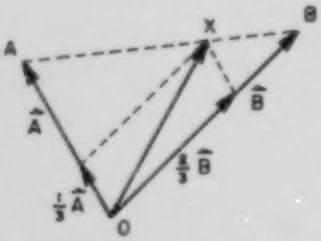
In step (2) $\frac{|\vec{B}|}{|\vec{A}|} = \frac{-r|\vec{P}|}{-r|\vec{Q}|} = \frac{|\vec{P}|}{|\vec{Q}|}$.

In step (5) , $d(O,D) = |rd(O,C)|$,

$$|\overrightarrow{D}| = |r\overrightarrow{C}| .$$

In step (6) , since the vectors are in opposite directions, $\overrightarrow{D} = r\overrightarrow{C}$.

110 When teaching this section, we would recommend that at first specific numbers be used for p and q . As an example, consider the line $\overleftrightarrow{AB} = \{X : \overrightarrow{X} = p\overrightarrow{A} + q\overrightarrow{B}$, where $p + q = 1\}$. Let $p = \frac{1}{3}$, $q = \frac{2}{3}$. Then $\overrightarrow{X} = \frac{1}{3}\overrightarrow{A} + \frac{2}{3}\overrightarrow{B}$.


A
X
B
$\overrightarrow{A}$
$\overrightarrow{B}$
$\frac{2}{3}\overrightarrow{B}$
$\frac{1}{3}\overrightarrow{A}$
O

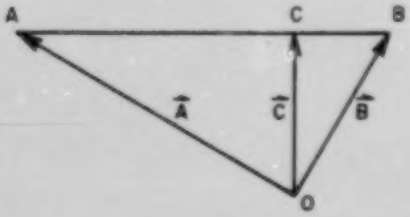
Take any vectors $\overrightarrow{A}$ and $\overrightarrow{B}$. Find the sum of $\frac{1}{3}\overrightarrow{A}$ and $\frac{2}{3}\overrightarrow{B}$ and verify, by construction, that X lies on $\overleftrightarrow{AB}$. Then let $p = \frac{4}{3}$ and $q = -\frac{1}{3}$ and see if the statement still holds.

Such experiences will help the students visualize what is really taking place.

111 In Chapter 2, a formula was developed for finding the coordinates of a point which divides a line segment in a given ratio. A comparable result for vectors is derived in Theorem 3-8. It may be of interest to the student to compare the derivations and the applications of the results.

Exercises 3-5

1.


A
C
B
$\overrightarrow{A}$
$\overrightarrow{C}$
$\overrightarrow{B}$
O

(a) if $\overrightarrow{A}$ is the zero vector, $\overrightarrow{C} = q\overrightarrow{B}$ and
if $\overrightarrow{B}$ is the zero vector $\overrightarrow{C} = p\overrightarrow{A}$

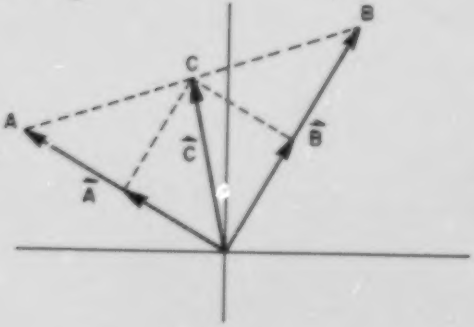
(b) if $\overrightarrow{C} = \overrightarrow{A}$, $p = 1$, $q = 0$

(c) (i) if $p > 0$, and $q > 0$, the terminal point of $\overleftarrow{C}$ lies in $\overline{AB}$.

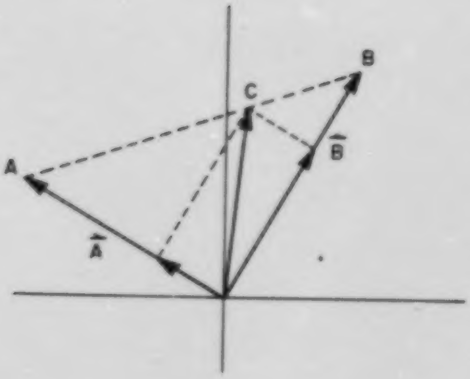
(ii) if $p < 0$, the terminal point of $\overleftarrow{C}$ lies on $\overleftrightarrow{AB}$ but not on $\overline{AB}$.

(iii) if $p = 0$, $\overleftarrow{C} = q\overleftarrow{B}$ and $\overleftarrow{C}$ lies on $\overleftrightarrow{OB}$.

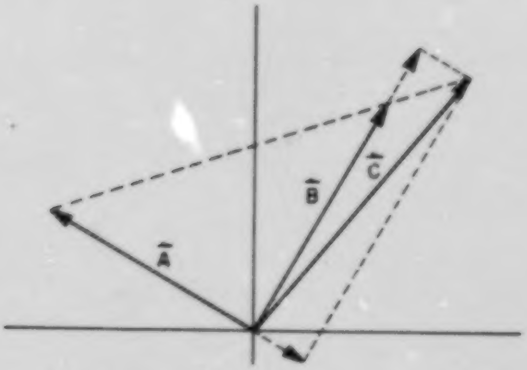
(d) (i) $p = q = \frac{1}{2}$


A
B
C
$\overleftarrow{A}$
$\overleftarrow{B}$
$\overleftarrow{C}$

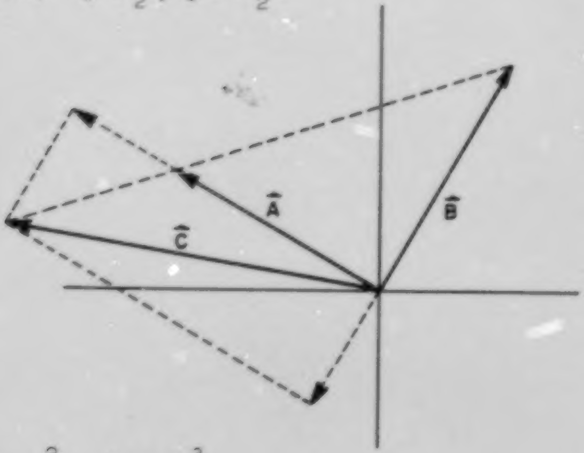
(ii) $p = \frac{1}{3}$, $q = \frac{2}{3}$


A
B
C
$\overleftarrow{A}$
$\overleftarrow{B}$

(iii) $p = -\frac{1}{4}$, $q = \frac{5}{4}$


$\vec{B}$
$\vec{C}$
$\vec{A}$

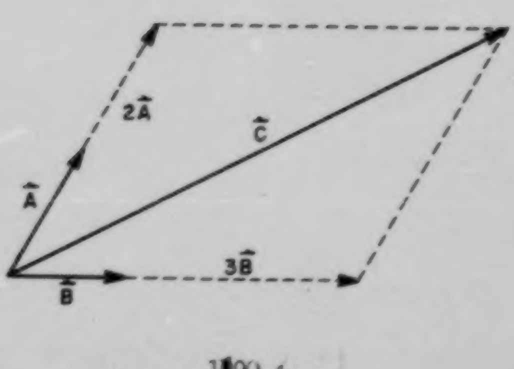
(iv) $p = \frac{3}{2}$, $q = -\frac{1}{2}$


$\vec{A}$
$\vec{B}$
$\vec{C}$

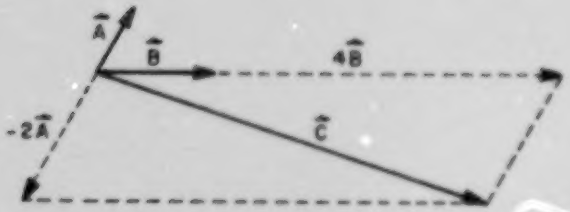
2. (a) $n = \frac{2}{5}$ and $m = \frac{3}{5}$

   (b) $m = \frac{5}{2}$ and $n = -\frac{3}{2}$

3. (a)


$2\vec{A}$
$\vec{C}$
$\vec{A}$
$3\vec{B}$
$\vec{B}$

(b)



4. Prove: $(r + s)\vec{P} = r\vec{P} + s\vec{P}$

   We note that $(r + s)\vec{P} \parallel r\vec{P} + s\vec{P}$

   Case 1: $r > 0$, $s > 0$.

   $r > 0$, $s > 0$ imply $r + s > 0$. Thus $(r + s)\vec{P}$ and $r\vec{P} + s\vec{P}$ have the same sense of direction, and

   $|(r + s)\vec{P}| = (r + s)|\vec{P}| = r|\vec{P}| + s|\vec{P}| = |r\vec{P}| + |s\vec{P}| = |r\vec{P} + s\vec{P}|$.

   Case 2: $r > 0$, $s < 0$. $r > |s|$

   Then $r + s > 0$ and $|(r + s)\vec{P}| = (r + s)|\vec{P}| = (r - |s|)|\vec{P}| = r|\vec{P}| - |s|\vec{P} = |r\vec{P}| - |s\vec{P}| = |r\vec{P} + s\vec{P}|$.

   Case 3: $r > 0$, $s < 0$, $r < |s|$

   Then $r + s < 0$ and $|(r + s)\vec{P}| = -(r + s)|\vec{P}| = (-|r| + |s|)|\vec{P}| = -|r|\vec{P} + |s|\vec{P} = -|r\vec{P}| + |s\vec{P}| = |r\vec{P} + s\vec{P}|$.

   Case 4: $r > 0$, $s < 0$, $r = |s|$

   $|(r + s)\vec{P}| = 0$ and $|r\vec{P} + s\vec{P}| = 0$

   Case 5: $r = 0$ or $s = 0$. The proof follows from the definition of scalar multiplication.

3-6. Components.

113 The notation introduced in this section simplifies vector manipulations. A component is itself a real number and not a vector.

What is actually done in this section is to establish an isomorphism between vectors with certain operations and ordered pairs of real numbers for which certain operations are defined. This leads eventually to vector spaces which are characterized abstractly by postulating the basic properties exhibited in this treatment. A set of postulates for a vector space can be found in SMSG Intermediate Mathematics, page 678-682 or any text on modern algebra or linear algebra.

Since the origin-vector is unique, the vector [a,b] equals the vector [c,d] if and only if a = c and b = d . This description of equality is used throughout the rest of the text and in many problems.

115 Part of the material presented earlier on the topic of linear combinations (See pages 108-109) is especially pertinent here. The unit vectors i = [1,0] and j = [0,1] in two dimensions and i = [1,0,0] , j = [0,1,0] and k = [0,0,1] in three dimensions are used in most applications of vector analysis. The i , j , k vectors are discussed in Chapter 8.

Exercises 3-6

1. (a) [7,3]
   (b) [-1,-1]
   (c) [20,24]
   (d) [-20,-24]
   (e) [-5,-6]
   (f) [-5,-6]
   (g) [10,9]
   (h) [14,-3]

2. (a) (1) [1,5]
   (2) [11,-8]
   (3) [13,-7]
   (4) [2,-16]
   (5) [12,-22]
   (6) [-10,76]

   (b) (1) $\overrightarrow{X} = \overrightarrow{A} + \overrightarrow{B} - \overrightarrow{C} = [0,-2]$

   (2) $\overrightarrow{X} = \frac{1}{5}(2\overrightarrow{A} + 3\overrightarrow{B} - 4\overrightarrow{C}) = [-1, -\frac{4}{5}]$

   (3) $\overrightarrow{X} = \overrightarrow{C} - \frac{2}{3}\overrightarrow{A} + \frac{2}{3}\overrightarrow{B} = [-\frac{2}{3}, \frac{31}{3}]$

   (4) $\overrightarrow{X} = \frac{1}{3}(\overrightarrow{B} + \overrightarrow{C} - \overrightarrow{A}) = [-\frac{2}{3}, \frac{14}{3}]$

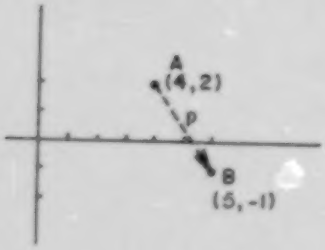
   (5) $\overrightarrow{X} = -2\overrightarrow{C} - 3\overrightarrow{B} = [-1,-24]$

   (6) $\overrightarrow{X} = -\frac{1}{3}\overrightarrow{A} - \frac{1}{2}\overrightarrow{B} = [-\frac{1}{2}, -\frac{4}{3}]$

3. 0

4. (a) $\sqrt{2}$
   (b) 5
   (c) $\sqrt{a^2 + b^2}$
   (d) 1

5.



$\overrightarrow{A} = 4i + 2j$

$\overrightarrow{B} = 5i - j$

$\overrightarrow{P} = \overrightarrow{B} - \overrightarrow{A} = (5-4)i + (-1-2)j$

$= i - 3j$

6. $\overrightarrow{O} = 0\cdot\overrightarrow{X} + 0\cdot\overrightarrow{Y}$

7. The midpoint of the line segment joining (2,5) and (5,8) is $(\frac{7}{2}, \frac{13}{2})$

$$\overrightarrow{p} = \frac{7}{2}i + \frac{13}{2}j$$

8. (a) $\overrightarrow{p} = \frac{\sqrt{3}}{2}i + \frac{1}{2}j$

(b) $\overrightarrow{q} = \frac{\sqrt{3}}{2}i - \frac{1}{2}j$

(c) $\overrightarrow{r} = \frac{4}{5}i - \frac{3}{5}j$

9. (a) $x = \frac{-13}{6}$ $y = \frac{23}{6}$

(b) $x = \frac{-1}{5}$ $y = \frac{4}{5}$

(c) $x = \frac{27}{13}$ $y = \frac{8}{13}$

(d) $x = r$ $y = \frac{-1-r}{2}$ for each real number. The real numbers form an infinite set.

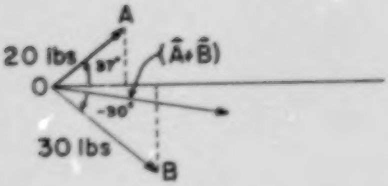
10. (a) $[a,b] = a[1,0] + b[0,1]$

(b) $[a,b] = \frac{a+b}{2}[1,1] + \frac{b-a}{2}[-1,1]$

(c) $[a,b] = -b\sqrt{2}\,[-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}] + (b-a)[-1,0]$

11. $T_x = 25\sqrt{3}$ lbs. ≈ 43.3 lbs. $T_y = 25$ lbs.

12. Letting 1 lb. correspond to 1 unit, set up a coordinate system.



$\overrightarrow{A} = [A_x, A_y] = [\,|\overrightarrow{A}| \cos 37^\circ, |\overrightarrow{A}| \sin 37^\circ]$

$= [20 \cdot \frac{4}{5}, 20 \cdot \frac{3}{5}]$

$\overrightarrow{B} = [B_x, B_y] = [\,|\overrightarrow{B}| \cos(-30^\circ), |\overrightarrow{B}| \sin(-30^\circ)]$

$= [30\frac{\sqrt{3}}{2}, 30(-\frac{1}{2})]$

$\overrightarrow{A} + \overrightarrow{B} = [16,12] + [15\sqrt{3}, -15] = [16 + 15\sqrt{3}, -3] \approx [42, -3]$.

13. (a) 24° , below x-axis in 4th quadrant. The components of the second vector, $\overrightarrow{B} = [26,-12]$

    (b) 32° from y-axis in 2nd quadrant. The components of the second vector, $\overrightarrow{B} = [-16,30]$

14. $24^\circ 30'$

15. (a) 21.3 lbs. acting 3° north of west.

    (b) 31.3 lbs. acting 2° north of west.

    In part (a) the components are $[-15\sqrt{2}, 15\sqrt{2} - 20]$

    In part (b) the components are $[-10 - 15\sqrt{2}, 15\sqrt{2} - 20]$

16. 14.6 lbs.

17. THEOREM 3-1. Let $\overrightarrow{P} = [a,b]$ $\overrightarrow{Q} = [c,d]$

    $\overrightarrow{P} + \overrightarrow{Q} = [a + c , b + d]$ and $\overrightarrow{Q} + \overrightarrow{P} = [c + a , b + d]$

    But addition in the real numbers is commutative so $a + c = c + a$, $b + d = d + b$. Therefore $[a + c , b + d] = [c + a, d + b]$ which means $\overrightarrow{P} + \overrightarrow{Q} = \overrightarrow{Q} + \overrightarrow{P}$.

    THEOREM 3-2. $\overrightarrow{P} = [a,b]$ $\overrightarrow{Q} = [c,d]$ $\overrightarrow{R} = [e,f]$

    $(\overrightarrow{P} + \overrightarrow{Q}) + \overrightarrow{R} = [(a + c) + e , (b + d) + f]$

    $\overrightarrow{P} + (\overrightarrow{Q} + \overrightarrow{R}) = [a + (c + e) , b + (d + f)]$

    But addition in the reals is associative which means

    $[(a + c) + e , (b + d) + f] = [a + (c + e) , b + (d + f)]$.

    Hence, $(\overrightarrow{P} + \overrightarrow{Q}) + \overrightarrow{R} = \overrightarrow{P} + (\overrightarrow{Q} + \overrightarrow{R})$.

    THEOREM 3-6. r and s are real numbers. $\overline{P} = [a,b]$, $\overline{R} = [c,d]$

    (1) $r(\overrightarrow{P} + \overrightarrow{Q}) = r([a + c , b + d])$

    $= [ra + rc , rb + rd]$

    $= [ra,rb] + [rc,rd]$

    $= r\overrightarrow{P} + r\overrightarrow{Q}$

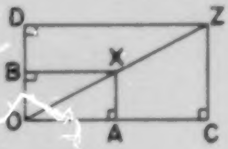
$$\begin{aligned}(2)\quad (r+s)\overset{\frown}{P} &= (r+s)[a,b]\\ &= [(r+s)a\,,\,(r+s)b]\\ &= [ra+sa\,,\,rb+sb]\\ &= [ra,rb]+[sa,sb]\\ &= r\overset{\frown}{P}+s\overset{\frown}{P}\end{aligned}$$

18. THEOREM 3-10. If $\overset{\frown}{X} = [a,b]$ and r is a real number, then $r\overset{\frown}{X} = [ra,rb]$.

Case 1: $a = 0$. Then $\overset{\frown}{X}$ lies along the y-axis. By definition, $r\overset{\frown}{X}$ lies along the y-axis also with terminal point at rb . So $r\overset{\frown}{X} = [r \cdot 0,\ rb] = [ra,rb]$.

Case 2: $b = 0$. By same argument $r\overset{\frown}{X} = [ra,rb]$.

Case 3: $a \neq 0$ and $b \neq 0$.
We get $\triangle OXA \sim \triangle OZC$
and $\triangle OXB \sim \triangle OZD$.



Let $\overset{\frown}{Z} = r\overset{\frown}{X}$.
$X = (a,b)$

So $\dfrac{d(O,X)}{d(O,Z)} = \dfrac{d(O,A)}{d(O,C)} = \dfrac{d(O,B)}{d(O,D)} = \dfrac{1}{r}$

But $d(O,A) = a \quad d(O,B) = b$
Therefore $d(O,C) = ra \quad d(O,D) = rb$ and $Z = (ra,rb)$.
(alternatively)
If $\overset{\frown}{X} = [a,b]$, define $\overset{\frown}{A} = [a,0]$ $\overset{\frown}{B} = [0,b]$ so that $\overset{\frown}{X} = A + B$.
$r\overset{\frown}{X} = r\overset{\frown}{A} + r\overset{\frown}{B}$
By Cases 1 and 2, $r\overset{\frown}{A} = [ra,0]$, $r\overset{\frown}{B} = [0,rb]$.
So $r\overset{\frown}{X} = [ra,0] + [0,rb] = [ra,rb]$.

19. The vector representation of each set below is written so that if $r = 0$ we obtain $\overset{\frown}{A}$ and if $r = 1$ we obtain $\overset{\frown}{B}$.

(a) $\{[2 - 6r\,,\,3 + 2r] : r$ is a real number$\}$
(b) $\{[1 + 2r\,,\,3 + 6r] : r$ is a real number$\}$
(c) $\{[4\,,\,-7 + 9r] : r$ is a real number$\}$
(d) $\{[2 + r] : r$ is a real number$\}$
(e) $\{[-3 + 4r\,,\,2 - 4r] : 0 \leq r \leq 1\}$
(f) $\{[1 + r] : 0 \leq r \leq 1\}$
(g) $\{[3 - 5r\,,\,4 - r] : 0 \leq r \leq 1\}$
(h) $\{[1 - 4r\,,\,-2 + 4r] : 0 \leq r\}$
(i) $\{[2 - r] : 0 \leq r\}$
(j) $\{[3 - 5r\,,\,4 - r] : 0 \leq r\}$
(k) $\{[-2 + 5r\,,\,3 + r] : 0 \leq r\}$

(l) $\{[2 - r] : 0 \leq r\}$

(m) $\{[3 - 5r, 4 - r] : 0 \geq r\}$

(n) $\{[-3 + 4r, 2 - 4r] : 0 < r < 1\}$

20. (a) $\overrightarrow{M} = [3,6]$, $\overrightarrow{T}_1 = [2,4]$, $\overrightarrow{T}_2 = [4,8]$

(b) $\overrightarrow{M} = [\frac{7}{2}, -\frac{9}{2}]$, $\overrightarrow{T}_1 = [\frac{4}{3}, -\frac{7}{3}]$, $\overrightarrow{T}_2 = [\frac{17}{3}, -\frac{20}{3}]$

(c) $\overrightarrow{M} = [\frac{a_1 + a_2}{2}, \frac{b_1 + b_2}{2}]$ $\overrightarrow{T}_1 = [\frac{2a_1 + b_1}{3}, \frac{2a_2 + b_2}{3}]$

$\overrightarrow{T}_2 = [\frac{a_1 + 2b_1}{3}, \frac{a_2 + 2b_2}{3}]$.

21. (a) $[2,8]$

(b) $[7]$

(c) $[0,0]$

(d) $[\frac{2}{3}, \frac{5}{2}]$

(e) $[\frac{39\pi + 2\sqrt{2}}{26(\sqrt{2} + \pi)}, \frac{26\pi + 24\sqrt{2}}{39(\sqrt{2} + \pi)}]$

(f) $[7]$

3-7. Inner Product.

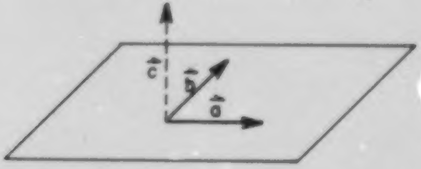
121 Although it is desirable algebraically to have some kind of vector multiplication, it is a little more difficult to introduce in a geometric framework. It would be possible to start by simply defining the inner product of two vectors by

$$[a_1, a_2] \cdot [b_1, b_2] = a_1 b_1 + a_2 b_2 .$$

This is quite satisfactory from the algebraic point of view, but does not connect very well with our development of vectors to this point. Hence a geometric approach is used by applying the law of cosines to the triangle formed by $\overrightarrow{X}$ and $\overrightarrow{Y}$. The definition of inner product is then made in terms of the resulting expression. The physical concept of work is one of the simplest applications of the inner product. It is included here to show that the inner product has relevance to a practical problem in science.

123 Theorem 3-13 establishes the connection between the geometric definition of inner product and its representation by components of the vectors. Either form can be used as indicated by a particular situation.

124 We did not present the vector product (or cross-product) $a \times b$ because some limitations had to be set for this chapter. The magnitude of $\overset{\frown}{a} \times \overset{\frown}{b} = |\overset{\frown}{a}||\overset{\frown}{b}| \sin \theta$; its direction lies along a line perpendicular to the plane determined by $\overset{\frown}{a}$ and $\overset{\frown}{b}$; and its sense of direction is determined by the motion of a right-hand screw when $\overset{\frown}{a}$ is rotated into $\overset{\frown}{b}$.



$\overset{\frown}{c} = \overset{\frown}{a} \times \overset{\frown}{b}$

You should note that $\overset{\frown}{a} \times \overset{\frown}{b} = -\overset{\frown}{b} \times \overset{\frown}{a}$ because the sense of direction is reversed. Thus the commutative law fails. $|\overset{\frown}{a} \times \overset{\frown}{b}|$ is the area of the parallelogram with $\overset{\frown}{a}$ and $\overset{\frown}{b}$ as sides.

Your interested students may like to investigate this topic in a standard text on vector analysis.

Exercises 3-7

1. (a) 0
   (b) 0
   (c) 1
   (d) 1
   (e) 0
   (f) -7
   (g) ac + bd

2. (a) -11
   (b) -66
   (c) 48
   (d) -110
   (e) 29
   (f) -205
   (g) -76
   (h) 0
   (i) 347
   (j) 64

3. (a) 90°
   (b) 80°
   (c) 109°
   (d) 60°
   (e) 132°
   (f) 34°
   (g) 0°
   (h) 180°

4. (a) $|\overset{\frown}{A}|^2 = 25$
   (b) $|\overset{\frown}{B}|^2 = 169$

5. (a) $\frac{-16}{3}$

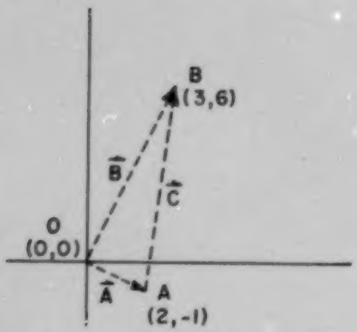
   (b) $\frac{16}{3}$

   (c) -3

   (d) 4

   (e) $-16i + 12j$; $16i - 12j$

6.



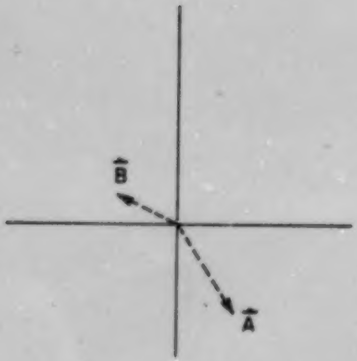
$\triangle AOB$ is a right $\triangle$.

If $\widehat{C}$ is as shown,

$\widehat{C} = \widehat{B} - \widehat{A}$

$\widehat{C} = i + 7j$

7.



$\widehat{A} = 2i - 3j$

$\widehat{B} = 2i + j$

$\widehat{A} \cdot \widehat{B} = |\widehat{A}| \cdot |\widehat{B}| \cos \theta$

(a) $\widehat{A} \cdot \widehat{B} = (2)(-2) + (-3)(1) = -7$

From $\widehat{A} \cdot \widehat{B} = |\widehat{A}||\widehat{B}| \cos \theta$, we find that $\cos \theta = -.863$

$\therefore$ **θ is approximately** 150°

(b) Since $\widehat{W} = \widehat{F} \cdot \widehat{S}$ and $\widehat{F} = \widehat{A} = 2i - 3j$

$\widehat{S} = \widehat{B} = 2i + 0j$,

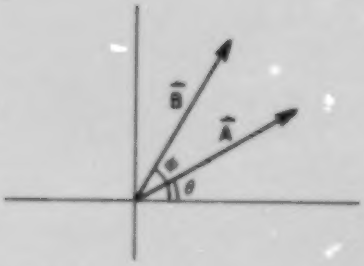
we have $\widehat{W} = \widehat{F} \cdot \widehat{S} = a_1 a_2 + b_1 b_2$, and

$W = (2)(2) + (-3)(0) = 4$ (in proper units).

8. (a) 940 ft. lbs.
   (b) 8660 ft. lbs.

9. (a) 10.6 ft.
   (b) 588.2 ft.

10.



(a) $|\hat{A}| = \cos^2\theta + \sin^2\theta = 1$, $|\hat{B}| = \cos^2\phi + \sin^2\phi = 1$, and $\hat{A}\cdot\hat{B} = |\hat{A}|\,|\hat{B}| \cos\psi$ where ψ is the angle between $\overline{A}$ and $\overline{B}$.

(b) In this case $\psi = \phi - \theta$
$\therefore\ \hat{A}\cdot\hat{B} = |\hat{A}|\,|\hat{B}| \cos(\phi - \theta) = 1 \cdot 1 \cdot \cos(\phi - \theta) = \cos(\phi - \theta)$.
Using components $\hat{A}\cdot\hat{B} = \cos\phi\cos\theta + \sin\phi\sin\theta$.
Thus $\cos(\phi - \theta) = \cos\phi\cos\theta + \sin\phi\sin\theta$.

11. To show $-1 \le \dfrac{\hat{X}\cdot\hat{Y}}{|\hat{X}|\cdot|\hat{Y}|} \le 1$.

This expression is defined only if $\hat{X} \ne \hat{0}$ and $\hat{Y} \ne \hat{0}$. In this case $\hat{X}\cdot\hat{Y}$ is defined as $|\hat{X}|\,|\hat{Y}| \cos\theta$. Now $-1 \le \cos\theta \le 1$ for any angle, XOY , $|\hat{X}|\,|\hat{Y}| \ne 0$ so we may multiply through by

$$1 = \frac{|\hat{X}|\,|\hat{Y}|}{|\hat{X}|\,|\hat{Y}|} \text{ getting } -1 \le \frac{\hat{X}\cdot\hat{Y}}{|\hat{X}|\,|\hat{Y}|} \le 1.$$

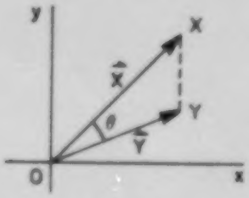
12. There is no associative law for inner products. The inner product of two vectors is a scalar.

3-8. Laws and Applications of the Inner (Dot) Product.

128 Most of the proofs of geometric theorems have been left for Chapter 4. These two proofs are given here to demonstrate that an abstract concept, such as the inner product of vectors, can be useful. The proof of the concurrence of the altitudes of a triangle is, we hope, impressive.

129 A bright student may ask why $\overleftrightarrow{BE}$ must intersect $\overleftrightarrow{CF}$ or why $\overleftrightarrow{AH}$ must intersect $\overleftrightarrow{BC}$. The answer is far from simple and involves a number of theorems involving the concepts of order, incidence, and betweenness. A careful treatment of such questions is given by E.E. Moise in his book Elementary Geometry from an Advanced Viewpoint. A careful non-vector proof of this theorem is in SMSG Geometry with Coordinates, p. 600-601.

131 A second derivation of the formula for the area of a triangle, $K = \frac{1}{2}|x_1y_2 - x_2y_1|$ is as follows:



(1) Consider $\triangle OXY$ and the realted non-zero vectors $\hat{X} = [x_1, x_2]$ and $\hat{Y} = [y_1, y_2]$ and the angle θ between them. Applying the trigonometric form for the area of a triangle, we have

$$K = \frac{1}{2}|\hat{X}||\hat{Y}| \sin\theta .$$

(2) Since $\hat{X}\cdot\hat{Y} = |\hat{X}||\hat{Y}| \cos\theta$, we have $|\hat{X}||\hat{Y}| = \dfrac{\hat{X}\cdot\hat{Y}}{\cos\theta}$, and

$$K = \frac{1}{2}(\hat{X}\cdot\hat{Y}) \tan\theta ,\ \theta \neq \frac{\pi}{2} .$$

(If the vectors are perpendicular, $K = \frac{1}{2}|\hat{X}||\hat{Y}|$)

(3) To write the result in terms of components, we observe the following:

(a) $\hat{X}\cdot\hat{Y} = x_1y_1 + x_2y_2$

(b) $\cos\theta = \dfrac{\hat{X}\cdot\hat{Y}}{|\hat{X}||\hat{Y}|} = \dfrac{x_1y_1 + x_2y_2}{\sqrt{x_1^2 + x_2^2}\ \sqrt{y_1^2 + y_2^2}}$

(c) $\sin\theta = \pm\sqrt{1 - \cos^2\theta} = \pm\sqrt{1 - \dfrac{(x_1y_1 + x_2y_2)^2}{(x_1^2 + x_2^2)(y_1^2 + y_2^2)}}$

$$= \frac{\pm(x_1y_2 - x_2y_1)}{\sqrt{x_1^2 + x_2^2}\ \sqrt{y_1^2 + y_2^2}} = \frac{\pm(x_1y_2 - x_2y_1)}{|X||Y|}$$

(4) Thus $K = \frac{1}{2}|x_1y_2 - x_2y_1|$.

Exercises 3-8,9

1. $\overset{\frown}{X} = [2,4]$ $\overset{\frown}{Y} = [-1,-3]$, $t = 5$

$$(t\overset{\frown}{X}) \cdot \overset{\frown}{Y} = t(\overset{\frown}{X} \cdot \overset{\frown}{Y}) = (\overset{\frown}{X}) \cdot (t\overset{\frown}{Y})$$

$$[10,20] \cdot [-1,-3] = 5([2,4] \cdot [-1,-3]) = [2,4] \cdot [-5,-15]$$

$$-10 - 60 = 5(-2 - 12) = -10 - 60$$

$$-70 = -70 = -70$$

2. If $\overset{\frown}{X} = [x_1,x_2]$ and $\overset{\frown}{Y} = [y_1y_2]$, prove that $(t\overset{\frown}{X}) \cdot \overset{\frown}{Y} = \overset{\frown}{X} \cdot (t\overset{\frown}{Y})$ for any scalar t .

Proof: $(t\overset{\frown}{X}) \cdot \overset{\frown}{Y} = \overset{\frown}{X} \cdot (t\overset{\frown}{Y})$ if

$[tx_1,tx_2] \cdot [y_1y_2] = [x_1,x_2][ty_1,ty_2]$ or

$tx_1y_1 + tx_2y_2 = tx_1y_1 + ty_1y_2$.

Since this last statement is true, the steps can be reversed to prove the original statement of the theorem.

3. To prove:

$\overset{\frown}{X} \cdot (a\overset{\frown}{Y} + b\overset{\frown}{Z}) = a(\overset{\frown}{X} \cdot \overset{\frown}{Y}) + b(\overset{\frown}{X} \cdot \overset{\frown}{Z})$, we note

that $\overset{\frown}{X} \cdot (a\overset{\frown}{Y}) + \overset{\frown}{X} \cdot (b\overset{\frown}{Z}) = a(\overset{\frown}{X} \cdot \overset{\frown}{Y}) + b(\overset{\frown}{X} \cdot \overset{\frown}{Z})$ (Theorem 3-14a)

and $a(\overset{\frown}{X} \cdot \overset{\frown}{Y}) + b(\overset{\frown}{X} \cdot \overset{\frown}{Z}) = a(\overset{\frown}{X} \cdot \overset{\frown}{Y}) + b(\overset{\frown}{X} \cdot \overset{\frown}{Z})$ (Theorem 3-14b)

4. (a) $(\overset{\frown}{A} + \overset{\frown}{B}) \cdot (\overset{\frown}{A} - \overset{\frown}{B}) = (\overset{\frown}{A} + \overset{\frown}{B}) \cdot \overset{\frown}{A} - (\overset{\frown}{A} + \overset{\frown}{B}) \cdot \overset{\frown}{B}$ (Theorem 3-14a)

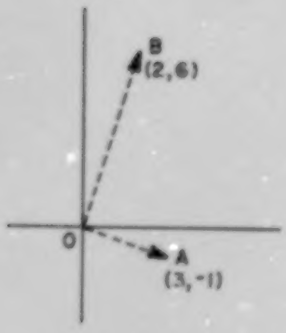
$= (\overset{\frown}{A} \cdot \overset{\frown}{A}) + (\overset{\frown}{B} \cdot \overset{\frown}{A}) - (\overset{\frown}{A} \cdot \overset{\frown}{B}) - (\overset{\frown}{B} \cdot \overset{\frown}{B})$ (Theorem 3-14a)

$= |\overset{\frown}{A}|^2 - |\overset{\frown}{B}|^2$ (Commutative Property of Inner Product and the fact that $\overset{\frown}{A} \cdot \overset{\frown}{A} = |\overset{\frown}{A}|^2$, $\overset{\frown}{B} \cdot \overset{\frown}{B} = |\overset{\frown}{B}|^2$)

(b) Construction: Two lines are parallel or intersect at a point.

(1) Theorem 3-12 and Theorem 3-14a .
(2) Same reason.
(3) Equality of real numbers and the commutative property.
(4) Additive property of equality.
(5) Theorem 3-14a and Theorem 12.
(6) $\overset{\frown}{a}$ lies on $\overset{\frown}{AD}$ and $(\overset{\frown}{c} - \overset{\frown}{b})$ lies on $\overset{\frown}{BC}$.

5.



$$K = \frac{1}{2}|x_1y_2 - x_2y_1|$$
$$= \frac{1}{2}|18 + 2|$$
$$= \frac{1}{2}|20| = 10$$

Check by alternate method : $\overline{OA} \perp \overline{OB}$ since $m_{\overline{OA}}$ is the negative reciprocal of $m_{\overline{OB}}$. $\therefore$ $\overline{OB}$ is an altitude of $\triangle OAB$.

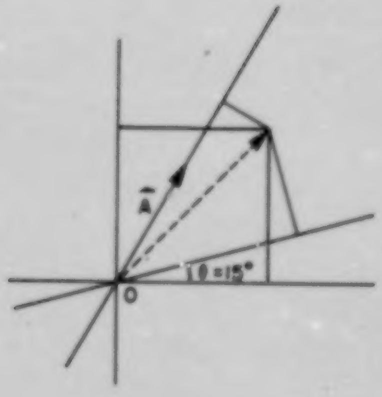
$d(O,A) = \sqrt{10}$ and $d(O,B) = \sqrt{40}$

$A = \frac{1}{2}(\sqrt{10})(\sqrt{40}) = 10$

6. (a) $-\frac{7\sqrt{5}}{5}$

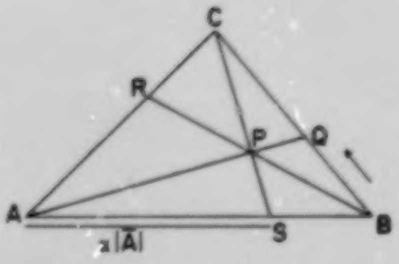
   (b) $-\frac{7\sqrt{13}}{13}$

7. (a) x direction, $15\sqrt{2}$
       y direction, $15\sqrt{2}$
   (b) 26.0
   (c) 29.4



CHALLENGE PROBLEMS

1. Let P be any point not on $\triangle ABC$. Let $\overleftrightarrow{AP}$, $\overleftrightarrow{BP}$, $\overleftrightarrow{CP}$ intersect sides $\overleftrightarrow{BC}$, $\overleftrightarrow{AC}$, $\overleftrightarrow{AB}$ respectively at points Q, R, S.



To show $\dfrac{d(A,S)}{d(S,B)} \cdot \dfrac{d(B,Q)}{d(Q,C)} \cdot \dfrac{d(C,R)}{d(R,A)} = 1$

Take origin at A.

Then $\vec{R} = \dfrac{d(A,R)}{d(A,C)}\vec{C}$, $\vec{S} = \dfrac{d(A,S)}{d(A,B)}\vec{B}$

$\overleftrightarrow{CS}$ contains points $x\vec{C} + (1 - x)\vec{S} = x\vec{C} + (1 - x)\dfrac{d(A,S)}{d(A,B)}\vec{B}$ (1)

$\overleftrightarrow{BR}$ contains points $y\vec{B} + (1 - y)\vec{R} = y\vec{B} + (1 - y)\dfrac{d(A,R)}{d(A,C)}\vec{C}$ (2)

For intersection $y = (1 - x)\dfrac{d(A,S)}{d(A,B)}$ $x = (1 - y)\dfrac{d(A,R)}{d(A,C)}$

which reduces to $x = \dfrac{d(A,R) \cdot d(S,B)}{d(A,B) \cdot d(A,C) - d(A,S) \cdot d(A,R)}$

$$y = \frac{d(A,S) \cdot d(R,C)}{d(A,B) \cdot d(A,C) - d(A,S) \cdot d(A,R)}$$

Thus $\vec{P} = \dfrac{d(A,S) \cdot d(R,C)}{d(A,B) \cdot d(A,C) - d(A,S) \cdot d(A,R)}\vec{B} +$

$$\frac{d(A,R) \cdot d(S,B)}{d(A,B) \cdot d(A,C) - d(A,S) \cdot d(A,R)}\vec{C}$$

But Q is on $\overleftrightarrow{AP}$, so for some t we have

$$t\vec{P} = \vec{B} + \frac{d(Q,B)}{d(B,C)}(\vec{C} - \vec{B})$$

whence $t\left(\dfrac{d(A,S) \cdot d(R,C)}{d(A,B) \cdot d(A,C) - d(A,S) \cdot d(A,R)}\right) = \dfrac{d(B,C) - d(Q,B)}{d(B,C)} = \dfrac{d(Q,C)}{d(B,C)}$ (3)

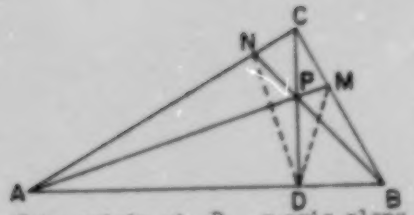
and $t\left(\dfrac{d(A,R) \cdot d(S,B)}{d(A,B) \cdot d(A,C) - d(A,S) \cdot d(A,R)}\right) = \dfrac{d(Q,B)}{d(B,C)}$ (4)

Substituting the expression for t obtained from (3) into (4) and simplifying we get

$$d(A,R) \cdot d(S,B) \cdot d(Q,C) \cdot d(B,C) \cdot d(B,C) \cdot d(A,S) \cdot d(R,C) \cdot d(Q,B)$$

which gives $\dfrac{d(A,S) \cdot d(C,R) \cdot d(Q,B)}{d(S,B) \cdot d(R,A) \cdot d(Q,C)} = 1$

2.



Consider $\triangle ABC$

$\overline{CD} \perp \overline{AB}$

P is a point on $\overline{CD}$

$\overleftrightarrow{PB}$ intersects $\overleftrightarrow{AC}$ at N

$\overleftrightarrow{PA}$ intersects $\overleftrightarrow{BC}$ at M

Take origin at D, x-axis along $\overleftrightarrow{AB}$, y-axis along $\overleftrightarrow{CD}$.

$A = [a,0]$ $B = [b,0]$ $C = [0,c]$ $P = [0,p]$

(This exercise considers only the case D strictly between A and B so that $a < 0 < b$ and $\frac{b}{a} \neq \frac{p}{c}$.)

If (x, y) is on $\overleftrightarrow{AC}$, then $y = \frac{c}{-a}(x - a)$

If (x, y) is on $\overleftrightarrow{PB}$, then $y = \frac{p}{-b}(x - b)$

Solving these to find coordinates of N we get

$$N = \left[\frac{ab(p - c)}{ap - bc}, \frac{cp(a - b)}{ap - bc}\right] = [N_x, N_y]$$

If (x,y) is on $\overleftrightarrow{BC}$, then $y = \frac{c}{-b}(x - b)$

If (x,y) is on $\overleftrightarrow{PA}$, then $y = \frac{p}{-a}(x - a)$

Solving for the coordinates of M we get

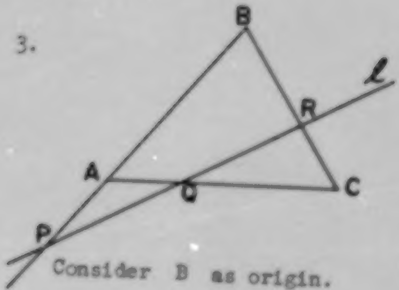
$$M = [\frac{ab(p - c)}{bc - ap}, \frac{ap(b - a)}{bc - ap}] = [M_x, M_y]$$

Because both $\angle NDC$ and $\angle MDC$ are smaller than 90° angles they are congruent if $|\sin \angle NDC|^2 = |\sin \angle MDC|$ for which it is enough that $|\sin \angle NDC|^2 = |\sin \angle MDC|^2$. But this follows from

$$|\sin \angle NDC|^2 = \frac{|N_x|^2}{d^2(ND)} = \frac{a^2b^2(c - p)^2}{(bc - ap)^2} \cdot \frac{(bc - ap)^2}{a^2b^2(c - p)^2 + c^2p^2(b - a)^2}$$

and $$|\sin \angle MDC|^2 = \frac{|M_x|^2}{d^2(MD)} = \frac{a^2b^2(p - c)^2}{(bp - ac)^2} \cdot \frac{(bc - ap)^2}{a^2b^2(c - p)^2 + c^2p^2(b - a)^2} .$$

3.



Let ℓ be any line which does not pass through any vertex of $\triangle ABC$. ℓ intersects $\overleftrightarrow{AB}$, $\overleftrightarrow{AC}$, $\overleftrightarrow{BC}$ at P, Q, R, respectively. (This contains implicit assumption that ℓ is parallel to none of the sides of $\triangle ABC$.)

Consider B as origin.

$$\vec{P} = \frac{d(B,P)}{d(B,A)}\vec{A} \qquad \vec{R} = \frac{d(B,R)}{d(B,C)}\vec{C}$$

Q is on $\overleftrightarrow{AC}$ so for some x, $\vec{Q} = x\vec{A} + (1 - x)\vec{C}$

Q is on $\overleftrightarrow{PR}$ so for some y, $\vec{Q} = y\vec{P} + (1 - y)\vec{R}$

$$= y\frac{d(B,P)}{d(B,A)} A + (1 - y)\frac{d(B,R)}{d(B,C)} C$$

Hence $x = y\,\dfrac{d(B,P)}{d(B,A)}$ $\qquad (1 - x) = (1 - y)\,\dfrac{d(B,R)}{d(B,C)}$

From these we get

$$\vec{Q} = \frac{d(B,P) \cdot d(B,C)}{d(B,C) \cdot d(B,P) - d(B,A) \cdot d(B,R)}\vec{A} + \frac{d(B,R) \cdot d(A,P)}{d(B,C) \cdot d(B,P) - d(B,A) \cdot d(B,P)}\vec{C} \tag{1}$$

$\vec{Q}$ is a defined point only if the denominator is not zero, which is the condition that excludes ℓ parallel to a side.

Similarly we may write

$$\vec{Q} = \frac{d(Q,C)}{d(A,C)}\vec{A} + \frac{d(Q,A)}{d(A,C)}\vec{C} \tag{2}$$

Then the coefficients of $\vec{A}$ and $\vec{C}$ in (1) must be equal respectively to the corresponding coefficients in (2). From which we find

$$\frac{d(A,Q)}{d(Q,C)} \cdot \frac{d(C,R)}{d(R,B)} \cdot \frac{d(B,P)}{d(P,A)} = 1$$

4. (a) To show $(x_1y_1 + x_2y_2)^2 \leq (x_1{}^2 + x_2{}^2)(y_1{}^2 + y_2{}^2)$

$$(x_1y_1 + x_2y_2)^2 = x_1{}^2y_1{}^2 + 2x_1y_1x_2y_2 + x_2{}^2y_2{}^2$$

$$(x_1{}^2 + x_2{}^2)(y_1{}^2 + y_2{}^2) = x_1{}^2y_1{}^2 + x_2{}^2y_1{}^2 + x_1{}^2y_2{}^2 + x_2{}^2y_2{}^2$$

Thus we need to show that

$$2x_1y_1x_2y_2 \leq x_1{}^2y_2{}^2 + x_2{}^2y_1{}^2$$

But this is true because we always have

$$(x_1y_2 - x_2y_1)^2 = x_1{}^2y_2{}^2 - 2x_1y_2x_2y_1 + x_2{}^2y_1{}^2 \geq 0$$

(b) Let $\overrightarrow{X} = [x_1, y_1]$, $\overrightarrow{Y} = [x_2, y_2]$ in 2-space.

Then we write $(\overrightarrow{X} \cdot \overrightarrow{Y})^2 \leq |\overrightarrow{X}|^2 \cdot |\overrightarrow{Y}|^2$

(c) $(\overrightarrow{X} \cdot \overrightarrow{Y})^2 = |\overrightarrow{X}|^2 \cdot |\overrightarrow{Y}|^2$ if and only if $x_1y_2 = x_2y_1$, that is, if and only if $\overrightarrow{X} = r\overrightarrow{Y}$, $r \neq 0$

Review Exercises

1. (a) $\overrightarrow{X} = \overrightarrow{A} + \overrightarrow{B} - \overrightarrow{C} = [0,-2]$

(b) $\overrightarrow{X} = \frac{1}{5}(2\overrightarrow{A} + 3\overrightarrow{B} - 4\overrightarrow{C}) = [-1, -\frac{4}{5}]$

(c) $\overrightarrow{X} = \overrightarrow{C} - \frac{2}{3}\overrightarrow{A} + \frac{2}{3}\overrightarrow{B} = [-\frac{2}{3}, \frac{31}{3}]$

(d) $\overrightarrow{X} = \frac{1}{3}(\overrightarrow{B} + \overrightarrow{C} - \overrightarrow{A}) = [-\frac{2}{3}, \frac{14}{3}]$

(e) $\overrightarrow{X} = -2\overrightarrow{C} - 3\overrightarrow{B} = [-1,-24]$

(f) $\overrightarrow{X} = -\frac{1}{3}\overrightarrow{A} - \frac{1}{3}\overrightarrow{B} = [-\frac{1}{2}, -\frac{4}{3}]$

2. Prove: $\overrightarrow{A} + \overrightarrow{X} = \overrightarrow{0}$ is satisfied by

$\overrightarrow{X} = (-1)\overrightarrow{A} = -\overrightarrow{A}$

Proof:

$\overrightarrow{A} + \overrightarrow{X} = \overrightarrow{A} + (-1\overrightarrow{A})$. (Substitution)

$= \overrightarrow{A} + -\overrightarrow{A}$ (Definition of $(-1)\overrightarrow{A}$)

$= \overrightarrow{0}$ ($-\overrightarrow{A}$ is additive inverse of $\overrightarrow{A}$)

3. $(rs)\overrightarrow{P} = r(s\overrightarrow{P})$

Proof: $(rs)\overrightarrow{P}$ and $r(s\overrightarrow{P})$ are parallel and have the same sense of direction.

$|(rs)\overrightarrow{P}| = |rs||\overrightarrow{P}| = |r||s||\overrightarrow{P}| = |r||s\overrightarrow{P}| = |r(s\overrightarrow{P})|$.

4. (a) $[14,-3]$
(b) $[-7,16]$
(c) $[-2,17]$
(d) $[6,0]$
(e) $[14,10]$
(f) $[-18,-4]$

5. (a) $[-6,-2]$
(b) $[\frac{17}{5}, -\frac{12}{5}]$
(c) $[-\frac{1}{3}, -\frac{1}{3}]$
(d) $[0, -\frac{2}{3}]$
(e) $[-7,0]$
(f) $[-\frac{13}{6}, 0]$

6. (a) 0
(b) 0
(c) 21
(d) -36
(e) 0
(f) -38
(g) 243
(h) -4
(i) -192
(j) -11

7. (a) $2\sqrt{13}$
(b) $2\sqrt{13} + 3\sqrt{10}$
(c) $2\sqrt{13} + 3\sqrt{10}$
(d) $-\sqrt{13}$
(e) $\sqrt{26}$
(f) $\sqrt{226}$
(g) $5\sqrt{13}$
(h) 0
(i) 36
(j) 329
(k) 225
(l) 26
(m) 105
(n) 52

8. (a) $2(2i + 3j) + 3(3i - 2j) - (-i + 3j) = 4i + 6j + 9i - 6j + i - 3j$
$= 14i - 3j$
(b) $-7i + 16j$
(c) $-2i + 17j$
(d) $6i$
(e) $14i + 10j$
(f) $-18i - 4j$

9. (a) $\vec{X} = 6i - 2j$

(b) $2(2i + 3j) + 3(3i - 2j) = 4(-i + 3j) + 5(x_1 i + x_2 j)$

$$4i + 6j + 9i - 6j = -4i + 12j + 5x_1 i + 5x_2 j$$

$$17i - 12j = 5x_1 i + 5x_2 j$$

$$5x_1 = 17$$

$$x_1 = \frac{17}{5}$$

$$5x_2 = -12$$

$$x_2 = -\frac{12}{5}$$

$$\vec{X} = \frac{17}{5} i - \frac{12}{5} j$$

(c) $\vec{X} = -\frac{1}{3} i - \frac{1}{3} j$

(d) $2i + 3j + 2(x_1 i + x_2 j) = 3i - 2j - i + 3j - x_1 i - x_2 j$

$$2i + 3j + 2x_1 i + 2x_2 j = 3i - 2j - i + 3j - x_1 i - x_2 j$$

$$0i + 2j = -3x_1 i - 3x_2 j$$

$$x_1 = 0$$

$$-3x_2 = 2$$

$$x_2 = -\frac{2}{3}$$

$$\vec{X} = -\frac{2}{3} j$$

(e) $\vec{X} = -7i$

(f) $\vec{X} = -\frac{13}{6} i$

10. (a) $(2i + 3j) \cdot (3i - 2j) = (2)(3) + (3)(-2) = 0$

(b) $2(2i + 3j) \cdot 3(3i - 2j) = (4i + 6j) \cdot (9i - 6j) = (4)(9) + (6)(-6) = 0$

(c) 21

(d) -36

(e) 0

(f) -38

(g) $\big(3(2i + 3j) + 5(3i - 2j)\big) \cdot \big(3(3i - 2j) - 2(-i + 3j)\big)$

$$\big(6i + 9j + 15i - 10j\big) \cdot \big(9i - 6j + 2i - 6j\big)$$

$$(21i - j) \cdot (11i - 12j) = (21)(11) + (-1)(-12) = 243$$

(h) -4

(i) -192

(j) 36

11. (a) $m\angle ABC = 90$ (in degrees)

$m\angle BCD = 100$

$m\angle CDA = 55$

$m\angle DAB = 115$

(b) Area of $\triangle OAB = 9$

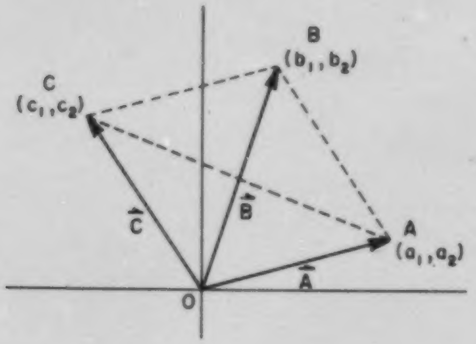
Area of $\triangle OBC = 8$

Area of $\triangle OAC = 7$

(c) Area of $\triangle ABC$ = Area of $\triangle OAB$ + Area of $\triangle OBC$ - Area $\triangle OAC$

$= 9 + 8 - 7 = 10$

12.



Area of $\triangle AOB = \frac{1}{2}|a_1b_2 - a_2b_1|$

Area of $\triangle BOC = \frac{1}{2}|b_1c_2 - b_2c_1|$

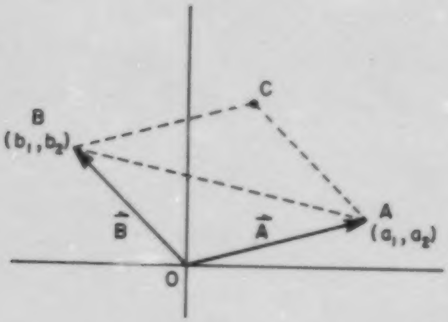
Area of $\triangle AOC = \frac{1}{2}|a_1c_2 - a_2c_1|$

From the diagram above: Area of $\triangle ABC$ = Area of $\triangle AOB$ + Area of $\triangle BOC$ - Area of $\triangle AOC$.

Area of $\triangle ABC = \frac{1}{2}|a_1b_2 - a_2b_1| + \frac{1}{2}|b_1c_2 - b_2c_1| - \frac{1}{2}|a_1c_2 - a_2c_1|$

Area of $\triangle ABC = \frac{1}{2}|a_1b_2 - a_2b_1 + b_1c_2 - b_2c_1 - a_1c_2 + a_2c_1|$

13.



Area of $\Delta AOB = \frac{1}{2}|a_1b_2 - a_2b_1|$

Area of $BOAC = 2(\text{Area of } \Delta AOB) = |a_1b_2 - a_2b_1|$

14. (a) $[-4,7]$
    (b) $[-4]$
    (c) $[\frac{1}{2}, -\frac{9}{2}, \frac{11}{2}]$
    (d) $[-15, \frac{23}{2}]$

15. (a) $\Delta ABC = \overline{AB} \cup \overline{BC} \cup \overline{CA}$

$$= \{(2 - 3r, 3 - r) : 0 \leq r \leq 1\} \cup \{[-1 + 2r, 2 + 2r] : 0 \leq r \leq 1\}$$
$$\cup \{[1 + r, 3 + r] : 0 \leq r \leq 1\}$$

Region $ABC = \{\vec{B} + r(\vec{A} - \vec{B}) + s(\vec{C} - \vec{B}) : 0 \leq r \leq 1, 0 \leq s \leq 1, r + s \leq 1\}$

$$= \{[-1 + 3r + 2s, 2 + r + 2s] : 0 \leq r \leq 1, 0 \leq s \leq 1, r + s \leq 1\}$$

Int.(Reg.ABC) $= \{\vec{B} + r(\vec{A} - \vec{B}) + s(\vec{C} - \vec{B}) : 0 < r < 1, 0 < s < 1, r + s < 1\}$

$$= \{[-1 + 3r + 2s, 2 + r + 2s] : 0 < r < 1, 0 < s < 1, r + s < 1\}$$

(b) $[1,3] = [-1 + 3(\frac{1}{2}) + 2(\frac{1}{4}), 2 + (\frac{1}{2}) + 2(\frac{1}{4})]$ where we certainly have

$$0 < r = \frac{1}{2} < 1, \ 0 < s = \frac{1}{4} < 1, \text{ and } r + s = \frac{3}{4} < 1$$

So $[3,1] \in$ Int.(Reg.ABC).

(c) $[1,1] = [-1 + 3r + 2s, 2 + r + 2s]$ if and only if $r = -\frac{3}{2}$, $s = -\frac{5}{4}$. So clearly $[1,1]$ does not satisfy the conditions to be in Region ABC .

(d) Segment $\overline{P_bP_c} = \{[1\ ,\ 1 + 2t] : 0 \leq t \leq 1\}$

From graphical considerations, we show $\overline{P_bP_c}$ intersects $\overline{AB}$ which is a subset of $\triangle ABC$. The conditions

$0 \leq r \leq 1\ ,\ 0 \leq t \leq 1\ ,\ [2 - 3r\ ,\ 3 - r] = [1\ ,\ 1 + 2t]$ are met for $t = \frac{5}{6}\ ,\ r = \frac{1}{3}$. Hence the segments intersect in the point $[1, \frac{8}{3}]$.

16. Region ABCD = Region BAD $\cup$ Region BDC $\cup$ Region BAC

$$= \{\bar{B} + r(\bar{A} - \bar{B}) + s(\bar{C} - \bar{B}) + t(\bar{D} - \bar{B}) : 0 \leq r \leq 1,\ 0 \leq s \leq 1,\ 0 \leq t \leq 1,\ r + s \leq 1,\ s + t \leq 1,\ r + t \leq 1\}$$

$$= \{[-1 + 3r + 2s + 3t,\ 2 + r + 2s + 2t] : 0 \leq r \leq 1,\ 0 \leq s \leq 1,\ 0 \leq t \leq 1,\ r + s \leq 1,\ s + t \leq 1,\ r + t \leq 1\}$$

Note: the commas indicate logical conjunction of the six individual conditions.

17. Region ABCD =

$$\{\bar{B} + r(\bar{A} - \bar{B}) + s(\bar{C} - \bar{B}) + t(\bar{D} - \bar{B}) : 0 \leq r \leq 1,\ 0 \leq s \leq 1,\ 0 \leq t \leq 1,\ r + s \leq 1,\ s + t \leq 1,\ r + t \leq 1\}$$

18. (a) 90°

(b) 97°

(c) 45°

(d) 61°

19. $\angle CAB = 90°$

$\angle ABC = 45°$

$\angle ACB = 45°$

20. $\angle PSR = 135°$

$\angle SRQ = 135°$

$\angle RQP = 45°$

$\angle QPS = 45°$

Trapezoid

Chapter 4

PROOFS BY ANALYTIC METHODS

This is the first of what some students refer to as "fun" chapters. There is nothing new to learn in the sense that there are no new theorems or definitions. The students have accumulated a variety of tools; now they will see how these tools may be used. In spite of the groans and complaints one hears from the class, most students thoroughly enjoy this type of thing.

Our primary concern in this chapter is that each student develop a systematic approach to solving problems by coordinates or vectors. We feel that a satisfactory beginning can be made by writing analytic proofs of familiar geometric theorems. It is also our aim that, while he is operating with these analytic tools, each student realize and appreciate the power available in the application of these tools. These methods represent a tremendous advance in mathematics, and the students should be aware of their heritage.

After a discussion of three methods of proof--by rectangular coordinates, by vectors, by polar coordinates--the chapter culminates in a section where the student must make a conscious choice of method. In order that the student not be denied this valuable opportunity to develop mathematical maturity, the teacher must avoid the temptation to decide for the student. Every student is entitled to learn what happens when he makes a poor choice. Furthermore, his choice may be, for him, the best.

The exercise solutions are given in the form we think is the most natural; but, to follow the spirit of the text, the teacher should accept any presentation which is mathematically sound. Then if the teacher feels that the student could have produced a simpler or more direct proof by using another method, this could be pointed out.

4-2. Proofs Using Rectangular Coordinates.

This section, which is concerned with proofs using rectangular coordinates, may be skimmed or swiftly reviewed if the class has already covered this material in another course. Some time might be saved in this way since the time allotment for this chapter assumes that most of the students have had little or no experience in this area.

The techniques we recommend are developed by means of examples. Following Example 1, we have suggested a short outline of systematic steps a student may follow for the problems which seem particularly suited to rectangular coordinates. To facilitate the study of the examples, we suggest that each student copy the figure and supply coordinates for it as the proof proceeds.

Among other things, Example 1 illustrates a rather delicate choice the student must make. On one hand, he must select coordinates which make the figure perfectly general; on the other hand, he should choose coordinates which make use of the information given in the problem. If he does this improperly, in the first instance he may have a proof which is valid for only a special case; in the second instance he may have a very complicated proof where a simple one would suffice. Example 1 shows how the choice of coordinates may be improved without losing generality in the figure.

142 We use the fact that $d(A,C) = d(B,C)$ to show that $\overline{CD}$ has no slope.

$$\sqrt{b^2 + c^2} = \sqrt{(b - 2a)^2 + c^2} ,$$

or

$$b^2 = b^2 - 4ab + 4a^2 .$$

Therefore,

$$4ab = 4a^2 ,$$

and, if $a \neq 0$, then $a = b$ and $\overline{CD}$ is vertical.

142 Regarding the choice of coordinates for A and B in Figure 4-4, we deliberately chose "-a" to the right of "a" so that some students who need the reminder may note that -a does not necessarily represent a negative number. It means the opposite of a; hence, when a is negative, -a is positive.

To show that C lies on the y-axis, we note that

$$d(A,C) = d(B,C) ,$$

or

$$\sqrt{(b - a)^2 + c^2} = \sqrt{(b - (-a))^2 + c^2} ,$$

or $$b^2 - 2ab + a^2 = b^2 + 2ab + a^2 .$$

Therefore, $$0 = 4ab,$$

and, if $a \neq 0$, then $b = 0$.

143 We justify the choice of abscissa for point C in Figure 4-5 in the following way. Let $D = (b,c)$ and $C = (d,c)$. Since $\overline{BC} || \overline{AD}$, their slopes are equal. Thus

$$\frac{c}{d - a} = \frac{c}{b} , \ (a \neq d) ,$$

and $$b = d - a ,$$

or $$d = a + b .$$

We are dealing with well-known and previously proved properties of geometric figures; therefore, some confusion may exist in the class as to which of these properties may be assumed in choosing coordinates for the figure. Although the teacher is at liberty, of course, to set up his own "ground rules", we recommend that only those properties ascribed to geometric figures by their definitions or by the hypothesis be allowed when selecting the coordinates. For the purposes of this section, we have also allowed the theorems (after proof) of Exercises 4-2. The teacher is not bound by this. Our reason for the exception is to make it unnecessary for a student to prove the same thing in two separate exercises.

144 To complete the proof of Example 3, we note that for the conclusion, $d(A,C) = d(B,C)$, to be true, we must have

$$\sqrt{4a^2 + 4c^2} = \sqrt{4b^2 + 4c^2} .$$

This will hold is $a^2 = b^2$. From the hypothesis, we have $d(A,N) = d(B,M)$, or

$$\sqrt{(b - 2a)^2 + c^2} = \sqrt{(2b - a)^2 + c^2} .$$

This simplifies to

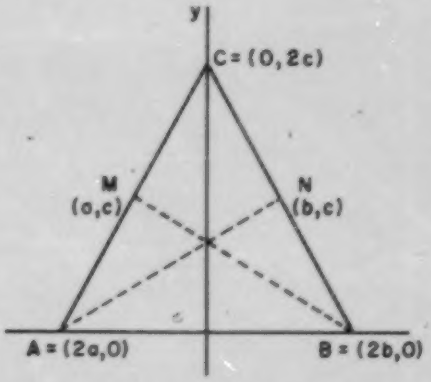


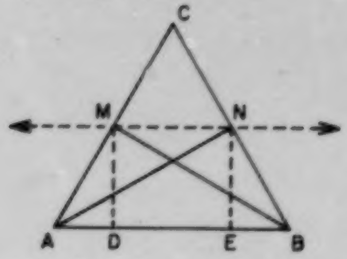
Figure 4-6

$$b^2 - 4ab + 4a^2 + c^2 = 4b^2 - 4ab + a^2 + c^2 ,$$

or $$3a^2 = 3b^2 ,$$

from which we have $a^2 = b^2$ as required.

144 We include a sample synthetic proof for Example 3:



Hypothesis:

$\overline{BM}$ and $\overline{AN}$ are medians.

$\overline{BM} \cong \overline{AN}$.

Conclusion:

$\overline{AC} \cong \overline{BC}$.

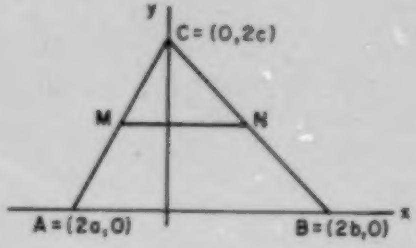
1.	$\overline{BM}$ and $\overline{AN}$ are medians.	Hypothesis.
2.	M is the midpoint of $\overline{AC}$; N is the midpoint of $\overline{BC}$.	Definition of median.
3.	$\overleftrightarrow{MN} \parallel \overleftrightarrow{AB}$.	The line joining the midpoints of two sides of a triangle is parallel to the line containing the third side.
4.	Introduce $\overline{MD}$ and $\overline{NE}$ perpendicular to $\overleftrightarrow{AB}$.	There is a unique perpendicular to a line from a point not on the line.
5.	$\overline{MD} \cong \overline{NE}$.	Parallels are everywhere equidistant.
6.	$\overline{BM} \cong \overline{AN}$.	Hypothesis.
7.	$\triangle BMD$ and $\triangle ANE$ are right triangles.	Perpendiculars form right angles.
8.	$\triangle BMD \cong \triangle ANE$.	Hypotenuse - leg theorem.
9.	$\angle DBM \cong \angle EAN$.	Corresponding angles of congruent triangles are congruent.
10.	$\overline{AB} \cong \overline{AB}$.	Reflective property of congruence for segments.
11.	$\triangle ABM \cong \triangle BAN$.	S. A. S. theorem.
12.	$\overline{AM} \cong \overline{BN}$.	Corresponding sides of congruent triangles are congruent.
13.	$d(A,M) = d(B,N)$.	Definition of congruence.
14.	$d(A,C) = d(B,C)$.	Definition of midpoint and multiplication property of equals.
15.	$\overline{AC} \cong \overline{BC}$.	Definition of congruence.

145 It is not anticipated that the teacher will assign all of the parts of Exercises 4-2 to a single student. The excess exercises may be used for test items. It is suggested that exercises 10, 13, 16 be assigned to everyone. These theorems are proved by vector methods in the next section, and the students may profit from a comparison of the two methods of proof.

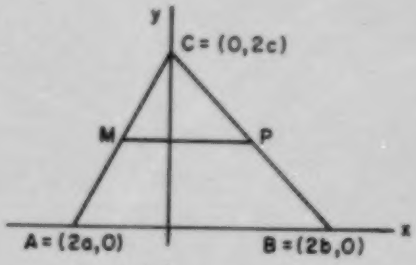
Exercises 4-2

(Note: Formal proofs are not presented here. We merely indicate the essentials of one possible solution for each problem.)

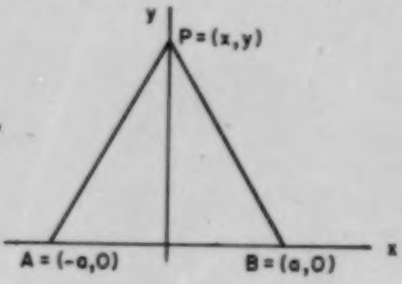
1. $M = (a,c)$; $N = (b,c)$.
   Slope of $\overline{MN} = 0$;
   slope of $\overline{AB} = 0$.
   $\therefore \overline{MN} \parallel \overline{AB}$.
   $d(M,N) = \sqrt{(a-b)^2} = |a-b|$.
   $d(A,B) = \sqrt{(2a-2b)^2} = |2a-2b|$
   $= 2|a-b|$.



2. $M = (a,c)$; since
   $\overline{MP} \parallel \overline{AB}$, $P = (x,c)$.
   P lies on $\overline{BC}$; therefore, slope of $\overline{PC}$ = slope of $\overline{BP}$; that is,
   $$\frac{-c}{x} = \frac{-c}{2b-x}.$$
   Thus, $x = b$ and $P = (b,c)$, the midpoint of $\overline{BC}$.



3. Part I. If $d(A,P) = d(B,P)$, then
   $\sqrt{(x+a)^2 + y^2} = \sqrt{(x-a)^2 + y^2}$,
   $x^2 + 2ax + a^2 + y^2 = x^2 - 2ax + a^2 + y^2$,
   and $4ax = 0$.
   Therefore, if $a \neq 0$, then $x = 0$ and P lies on the y-axis, the perpendicular bisector of $\overline{AB}$.



   Part II. If P lies on the perpendicular bisector of $\overline{AB}$, then $x = 0$ and $d(A,P) = \sqrt{a^2 + y^2} = \sqrt{(-a)^2 + y^2} = d(B,P)$.

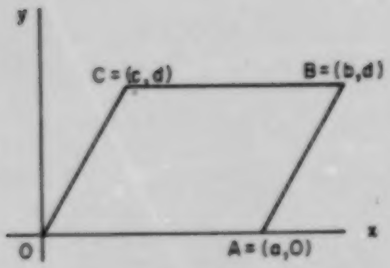
4. By definition $\overline{OC} \| \overline{AB}$ and their slopes are equal. Thus

$$\frac{d}{c} = \frac{d}{b - a}, \quad (a \neq b),$$

and $b = a + c$. Therefore,

$d(B,C) = \sqrt{a^2} = |a| = d(A,O)$

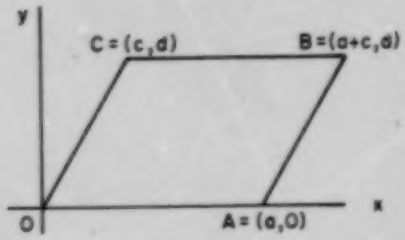
and $d(C,O) = \sqrt{c^2 + d^2} = d(B,A)$.



5. $B = (a + c,d)$ because $d(B,C) = d(O,A)$ and $\overline{BC} \| \overline{OA}$.

Slope of $\overline{OC} = \frac{d}{c} =$ slope of $\overline{AB}$;

therefore, $\overline{OC} \| \overline{AB}$.



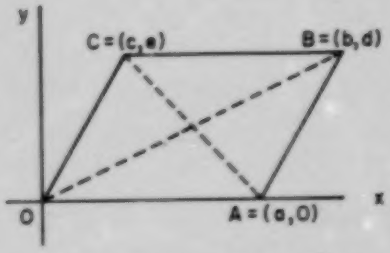
6. Midpoint of $\overline{OB} = \left(\frac{b}{2}, \frac{d}{2}\right)$;

midpoint of $\overline{AC} = \left(\frac{a + c}{2}, \frac{e}{2}\right)$.

Since $\left(\frac{b}{2}, \frac{d}{2}\right) = \left(\frac{a + c}{2}, \frac{e}{2}\right)$,

$b = a + c$ and $d = e$.

This satisfies the conditions for the theorem of Exercise 5.



7. Since OABC is a parallelogram, it may have coordinates as in Exercise 5. Since $d(O,B) = d(A,C)$,

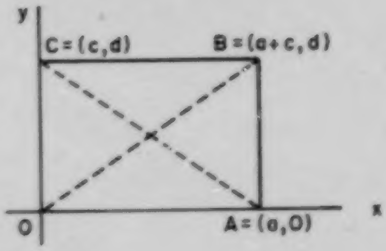
$$\sqrt{(a + c)^2 + d^2} = \sqrt{(a - c)^2 + d^2},$$

$$a^2 + 2ac + c^2 + d^2 = a^2 - 2ac + c^2 + d^2,$$

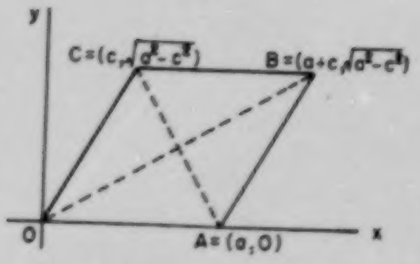
and $4ac = 0$.

If $a \neq 0$, then $c = 0$ and $B = (a,d)$;

therefore, $\angle OAB$ is a right angle.



8. The coordinates shown in the figure take account of the fact that a rhombus is a parallelogram with congruent sides.

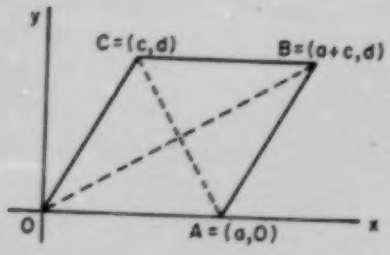

y
C=(c,√(a²-c²))
B=(a+c,√(a²-c²))
O
A=(a,0)
x

The slope of $\overline{AC}$ is $\dfrac{\sqrt{a^2 - c^2}}{c - a}$;

the slope of $\overline{OB}$ is $\dfrac{\sqrt{a^2 - c^2}}{a + c}$.

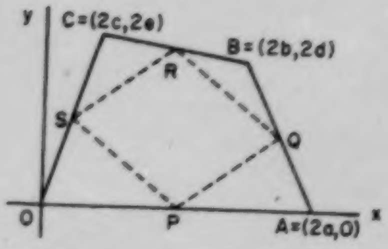
The product of the slopes is $\dfrac{a^2 - c^2}{c^2 - a^2} = -1$; hence, the diagonals are perpendicular.

9. The slope of $\overline{AC} = \dfrac{d}{c - a}$; the slope of $\overline{OB} = \dfrac{d}{a + c}$. Since $\overline{AC} \perp \overline{OB}$, $\dfrac{d}{c - a} \cdot \dfrac{d}{a + c} = -1$.


y
C=(c,d)
B=(a+c,d)
O
A=(a,0)
x

Therefore, $d^2 = a^2 - c^2$, or $a^2 = c^2 + d^2$. Hence, $|a| = \sqrt{c^2 + d^2} = d(O,C) = d(O,A)$.

10. $P = (a,0)$; $Q = (a + b,d)$; $R = (b + c,d + e)$; $S = (c,e)$.


y
C=(2c,2e)
B=(2b,2d)
R
S
Q
O
P
A=(2a,0)
x

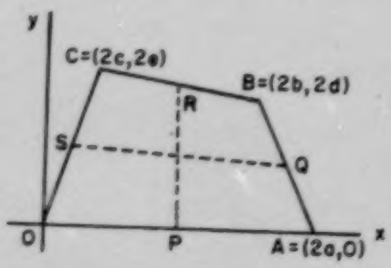
Slope of $\overline{PQ}$ = slope of $\overline{RS} = \dfrac{d}{b}$;

slope of $\overline{PS}$ = slope of $\overline{RQ} = \dfrac{e}{c - a}$.

11. $P = (a,0)$; $Q = (a + b,d)$; $R = (b + c,d + e)$; $S = (c,e)$.

Midpoint of $\overline{RP} = \left(\dfrac{a + b + c}{2}, \dfrac{d + e}{2}\right)$;

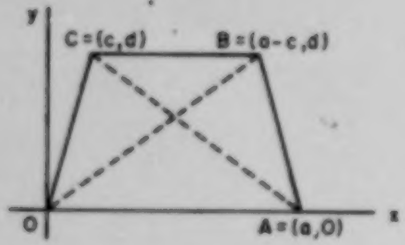
midpoint of $\overline{SQ} = \left(\dfrac{a + b + c}{2}, \dfrac{d + e}{2}\right)$.


y
C=(2c,2e)
B=(2b,2d)
R
S
Q
O
P
A=(2a,0)
x

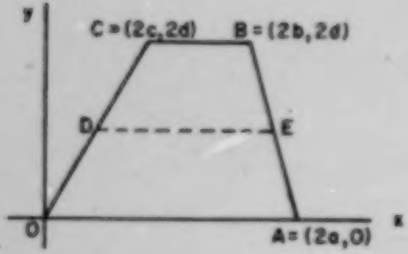
12. $d(A,C) = \sqrt{(c - a)^2 + d^2}$

$= \sqrt{(a - c)^2 + d^2}$

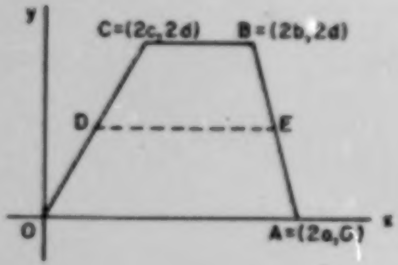
$= d(O,B)$.


y
C = (c,d)
B = (a-c,d)
O
A = (a,0)
x

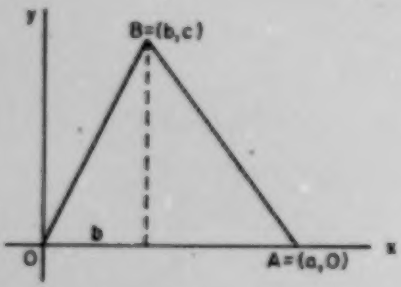
13. $D = (c,d)$; $E = (a + b,d)$.
Slope of $\overline{DE} = 0 =$ slope of $\overline{OA}$
and slope of $\overline{BC}$.
$d(O,A) = d(C,B) = 2a + 2b - 2c$
$= 2(a + b - c)$.
$d(D,E) = a + b - c$.


y
C = (2c,2d)
B = (2b,2d)
D
E
O
A = (2a,0)
x

14. $D = (c,d)$; let $E = (e,d)$.
Since E lies on $\overline{AB}$, the slope
of $\overline{BE}$ = the slope of $\overline{AE}$; hence,
$\frac{d}{2b - e} = \frac{d}{e - 2a}$, $2e = 2a + 2b$,
and $e = a + b$. Therefore
$E = (a + b,d)$, the midpoint of $\overline{AB}$.


y
C = (2c,2d)
B = (2b,2d)
D
E
O
A = (2a,0)
x

15. Let the acute angle be at O.
$(d(A,B))^2 = (b - a)^2 + c^2$
$= b^2 - 2ab + a^2 + c^2$.
Also $(d(O,B))^2 + (d(O,A))^2$
$- 2d(O,A)b$
$= (b^2 + c^2) + a^2 - 2ab$
$= b^2 - 2ab + a^2 + c^2$.

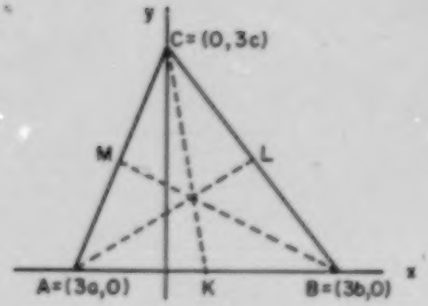

y
B = (b,c)
b
O
A = (a,0)
x

16. $K = \left(\frac{3a + 3b}{2}, 0\right)$;

$L = \left(\frac{3b}{2}, \frac{3c}{2}\right)$;

$M = \left(\frac{3a}{2}, \frac{3c}{2}\right)$.

The point $(a + b, c)$ divides each of $\overline{CK}$, $\overline{BM}$, and $\overline{AL}$ in the ratio 2:1.

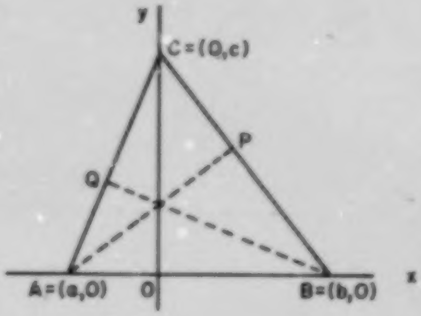


17. Since $\overline{AP} \perp \overline{BC}$, the slope of $\overline{AP} = \frac{b}{c}$; since $\overline{BQ} \perp \overline{AC}$, the slope of $\overline{BQ} = \frac{a}{c}$.

$\overleftrightarrow{AP} = \{(x,y): \ y = \frac{b}{c}(x - a)\}$;

$\overleftrightarrow{BQ} = \{(x,y): \ y = \frac{a}{c}(x - b)\}$.

Since the intersection must lie on the y-axis, $x = 0$, and the point is $(0, -\frac{ab}{c})$.



* 18. In the solution of this exercise we wish to make use of the proposition: The segment joining the center of a circle to the midpoint of a chord of the circle is perpendicular to the chord. We dispose of this proposition first.

Since $d(O,A) = d(O,B)$,

$\sqrt{4a^2 + 4c^2} = \sqrt{4b^2 + 4d^2}$,

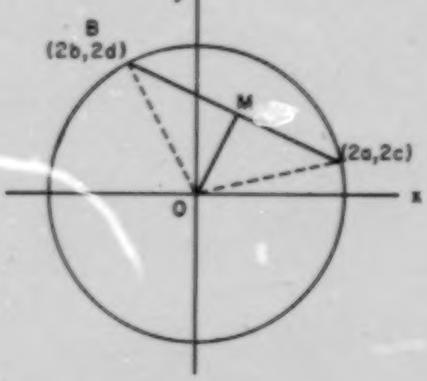
or $a^2 + c^2 = b^2 + d^2$.

The slope of $\overline{AB} = \frac{c - d}{a - b}$;

the slope of $\overline{OM} = \frac{c + d}{a + b}$.

The product of these slopes is

$\frac{c^2 - d^2}{a^2 - b^2}$, and, since

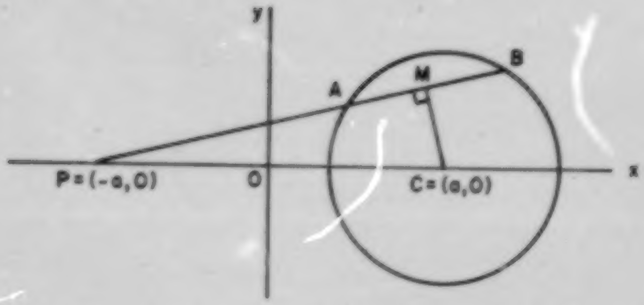


$a^2 + c^2 = b^2 + d^2$, $c^2 - d^2 = b^2 - a^2$.

Substituting in the product of the slopes obtained above, we have

$$\frac{b^2 - a^2}{a^2 - b^2} = -1 \ ;$$

therefore, $\overline{OM} \perp \overline{AB}$.



We return to the first problem and select a coordinate system as depicted in the figure. We have placed the origin at the midpoint of $\overline{PC}$, and we let $M = (x,y)$.

We then have $d(P,M) = \sqrt{(x + a)^2 + y^2}$,

$d(M,C) = \sqrt{(x - a)^2 + y^2}$,

and $d(P,C) = 2a$.

By employing the Pythagorean Theorem in $\triangle PCM$ we obtain

$$(x + a)^2 + y^2 + (x - a)^2 + y^2 = 4a^2 \ ,$$

$$x^2 + 2ax + a^2 + y^2 + x^2 - 2ax + a^2 + y^2 = 4a^2 \ ,$$

$$2x^2 + 2y^2 = 2a^2 \ ,$$

or

$$x^2 + y^2 = a^2 \ .$$

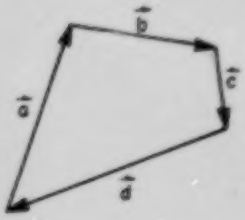
We recognize this as an equation of the circle of radius a which has its center at the origin. However, the entire circle is not the locus in the case we have depicted. The locus is the arc of this circle which is contained in or on the fixed circle. This is the case for which the radius, r , of the fixed circle is less than $2a$; the point P is exterior to the fixed circle. If $r = 2a$, P is on the fixed circle; if $r > 2a$, P is inside the fixed circle. In both of these latter two cases, the entire circle $x^2 + y^2 = a^2$ is the locus.

4-3. Proofs Using Vectors.

The purpose of this section is to show another method of proving geometric porpositions. It is inappropriate to say that one method is superior to another. For a particular problem, one method may be simpler than another method, but the point here is to increase the diversity of available methods. Using vectors may be an approach which, though new to many students can be of considerable interest to them. If the teacher (or any student) wishes to pursue this topic of vectors applied to geometry, he may consult Elementary Vector Geometry by Seymour Schuster.

147 A reference to the discussion of Figure 3-8 in Chapter 3 may help some students to understand the vector addition performed in Example 1. This example is Exercise 13 of the preceding set.

An application of vector addition which may interest some students involves the sum around a closed region. For example, $\vec{a} + \vec{b} + \vec{c} + \vec{d} = \vec{0}$. One of Kirchhoff's Laws, which is widely used in dealing with electrical circuits, states that the sum of the potential (voltage) drops around a closed circuit is zero.



147 The students should discover that altering the directions of any of the vectors in Figure 4-8 will not essentially change the proof--only some details will be modified. The students may encounter some difficulty, however, if they are careless in the way they label the vectors. For example, since E is the midpoint of $\overline{AD}$ and we chose $\vec{a}$ to designate the vector from A to E, the vector from E to D is also labeled $\vec{a}$. But if we used the vector from D to E, it would be labeled $-\vec{a}$.

148 Example 2 is Exercise 10 of Exercises 4-2. We have suggested to the student that he copy Figure 4-9. We should like to emphasize this suggestion. We think this will help the student to see that the choice of an origin is completely arbitrary, and the drawing of the origin-vectors as the proof proceeds may aid in visualizing the steps of the proof.

149 Example 3 is Exercise 16 of Exercises 4-2. Note that a particular choice of origin (aided by a prior knowledge of the result) greatly simplifies the proof.

In solving any sort of problem it is difficult in general to tell beforehand what will "work" and what will not. This is true of the more complicated exercises where a particular choice of the roigin may give simpler calculations than occur with another choice. In general, an origin should be selected which allows the hypothesis to be expressed simply. It should also be chosen so that the number of independent vectors needed is as small as possible. Apart from this, experience gained from trial and error is a valuable help. If calculations bog down with one choice, perhaps another choice should be made. However, some propositions simply do not possess short, elegant proofs.

151 The centroid of an area or a volume can be defined in mathematical terms using integral calculus. The center of gravity of a thin uniform sheet or of a uniform mass is the centroid of the corresponding mathematical area or volume.

Physically, the center of gravity of an object will always lie on a vertical line through a point of suspension of the object. Thus the center of gravity of a triangular object can also be determined experimentally by suspending it from 2 different points, say 2 vertices, and then determining where the lines of suspension intersect.

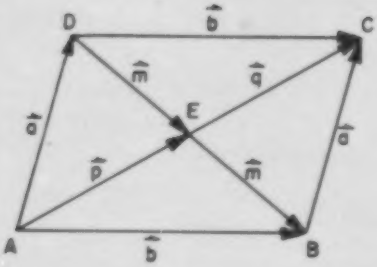
151 There may be some mystery surrounding the choice of unit vectors in Example 4. Of course, we always can say, "It works!" But we can give a more sound justification. The fact that we need an angle bisector could lead someone to think of the diagonals of a rhombus, and the congruent sides of a rhombus could lead someone to think of unit vectors. Students (and teachers) should not be discouraged if they do not think of things like this; years of experience and/or a little luck play a large part in these activities.

153 Exercises 5 and 6 of Section 4-3 are the same theorems used in Examples 3 and 1 of Section 4-2. These may be assigned for purposes of comparing the two methods of proof.

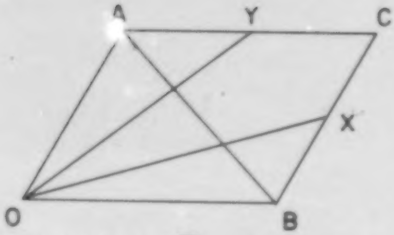
Exercises 4-3

(Note: Formal proofs are not presented here. We merely indicate the essentials of one possible solution for each problem.)

1. Let E be the midpoint of $\overline{DB}$. We have $\vec{p} = \vec{a} + \vec{m}$ and $\vec{q} = \vec{m} + \vec{a}$; therefore, $\vec{p} = \vec{q}$ and point E bisects $\overline{AC}$.



2\.



Consider the diagram at the left.
$\overline{AY} \cong \overline{YC}$ $\quad \overline{CX} \cong \overline{XB}$
We wish to show that $\overline{OY}$ and $\overline{OX}$ trisect $\overline{AB}$, and that $\overline{AB}$ passes through points of trisection of $\overline{OY}$ and $\overline{OX}$.

Any point on $\overline{AB}$ can be represented by $z\vec{A} + (1 - z)\vec{B}$, $0 \leq z \leq 1$.
Any point on $\overline{OY}$ can be represented by $y\vec{Y}$, $0 \leq y \leq 1$.
Any point on $\overline{OX}$ can be represented by $x\vec{X}$, $0 \leq x \leq 1$.
We wish to find values of x and z such that $z\vec{A} + (1 - z)\vec{B} = x\vec{X}$.

But we also know $\vec{X} = \frac{1}{2}(\vec{C} + \vec{B})$ and $\vec{C} = \vec{A} + \vec{B}$
so we want $z\vec{A} + (1 - z)\vec{B} = \frac{1}{2}x(\vec{A} + \vec{B} + \vec{B})$

$$z\vec{A} + (1 - z)\vec{B} = \frac{1}{2}x\vec{A} + x\vec{B}$$

so we find $z = \frac{1}{3}$ $\quad x = \frac{2}{3}$

Thus the intersection is at $\frac{1}{3}\vec{A} + \frac{2}{3}\vec{B} = \frac{2}{3}\vec{X}$

We find by similar computations that $\overline{AB}$ intersects $\overline{OY}$ at $\frac{2}{3}\vec{A} + \frac{1}{3}\vec{B} = \frac{2}{3}\vec{Y}$
This means $\overline{OY}$ and $\overline{OX}$ trisect $\overline{AB}$ and also that $\overline{AB}$ passes through points of trisection of $\overline{OX}$ and $\overline{OY}$.

3. Using A as the origin, we have

$\overrightarrow{P} = \frac{1}{2}(\overrightarrow{B} + \overrightarrow{C})$,

$\overrightarrow{Q} = \frac{1}{2}\overrightarrow{C}$,

$\overrightarrow{R} = \frac{1}{2}\overrightarrow{B}$.

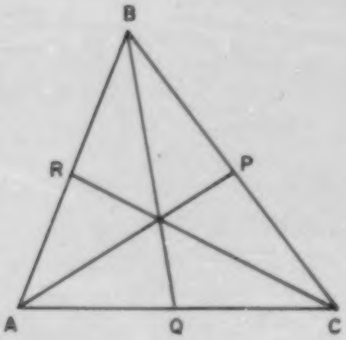


Figure 4-12

The intersection of medians $\overline{BQ}$ and $\overline{CR}$ can be located by finding the values of x and y which solve

$x\overrightarrow{B} + (1 - x)\overrightarrow{Q} = y\overrightarrow{C} + (1 - y)\overrightarrow{R}$.

Substituting, we obtain

$x\overrightarrow{B} + \frac{1}{2}\overrightarrow{C} - \frac{1}{2}x\overrightarrow{C} = y\overrightarrow{C} + \frac{1}{2}\overrightarrow{B} - \frac{1}{2}y\overrightarrow{B}$.

Equating corresponding coefficients, we have

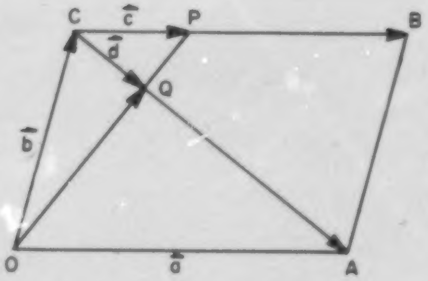
$$x = \frac{1}{2}(1 - y) \quad \text{and} \quad y = \frac{1}{2}(1 - x),$$

from which we obtain $x = y = \frac{1}{3}$.

This tells us that the intersection of $\overline{BQ}$ and $\overline{CR}$ is $\frac{1}{3}(\overrightarrow{B} + \overrightarrow{C})$, which is a trisection point of each of these medians. A trisection point of $\overline{AP}$ is

$$\frac{2}{3}\overrightarrow{P} = \frac{2}{3} \cdot \frac{1}{2}(\overrightarrow{B} + \overrightarrow{C}) = \frac{1}{3}(\overrightarrow{B} + \overrightarrow{C}) .$$

4. Since $\frac{d(C,P)}{d(C,B)} = \frac{1}{r}$, the vector from C to P is $\overrightarrow{c} = \frac{1}{r}\overrightarrow{a}$.



The vector from C to A is $(\overrightarrow{a} - \overrightarrow{b})$, and we wish to find $n(\overrightarrow{a} - \overrightarrow{b}) = \overrightarrow{d}$, the scalar multiple of it. The vector from O to Q may be expressed as $(\overrightarrow{b} + \overrightarrow{d})$ or as a scalar multiple of the vector

$\vec{p}$. We therefore have

$$\vec{b} + \vec{d} = m(\vec{b} + \vec{c}),$$

$$\vec{b} + n(\vec{a} - \vec{b}) = m(\vec{b} + \frac{1}{r}\vec{a}),$$

$$\vec{b} + n\vec{a} - n\vec{b} = m\vec{b} + \frac{m}{r}\vec{a}.$$

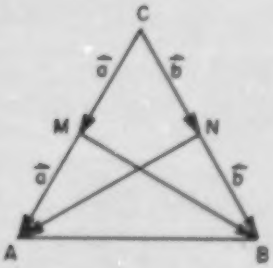
Equating corresponding coefficients gives us

$$n = \frac{m}{r} \text{ and } m = (1 - n);$$

for these equations we find $n = \frac{1}{r + 1}$. Therefore,

$$\vec{d} = \frac{1}{r + 1}(\vec{a} - \vec{b}), \text{ and } \frac{d(C,Q)}{d(C,A)} = \frac{1}{r + 1}.$$

5. From the diagram we see that the vector from N to A is $2\vec{a} - \vec{b}$ and the vector from M to B is $2\vec{b} - \vec{a}$. Since $d(N,A) = d(M,B)$, we have $|2\vec{a} - \vec{b}| = |2\vec{b} - \vec{a}|$. Using the Law of Cosines, we may write this as



$$\sqrt{4|\vec{a}|^2 + |\vec{b}|^2 + 4\vec{a}\cdot\vec{b}} = \sqrt{4|\vec{b}|^2 + |\vec{a}|^2 + 4\vec{b}\cdot\vec{a}}.$$

This equation simplifies to

$$4|\vec{a}|^2 + |\vec{b}|^2 = 4|\vec{b}|^2 + |\vec{a}|^2,$$

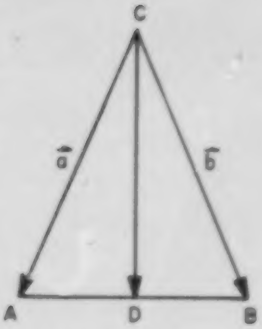
or

$$3|\vec{a}|^2 = 3|\vec{b}|^2.$$

From this we see that $2|\vec{a}| = 2|\vec{b}|$, and $\triangle ABC$ is isosceles.

This vector proof of Example 3, Section 4-2, is somewhat artificial because of the use of the Law of Cosines. It may be profitable for the students to compare this proof with the rectangular coordinate and synthetic proofs appearing in Section 4-2 of this commentary. It can be noted that applying vectors to equal lengths may become awkward if the vectors are not parallel.

6. The vector from C to D may be expressed as $\frac{1}{2}\vec{a} + \frac{1}{2}\vec{b}$, and the vector from A to B may be expressed as $\vec{b} - \vec{a}$. The product of these two vectors is



$$(b - a)\cdot\left(\frac{1}{2}\vec{a} + \frac{1}{2}\vec{b}\right)$$

$$= \frac{1}{2}\vec{a}\cdot\vec{b} - \frac{1}{2}\vec{a}\cdot\vec{a} + \frac{1}{2}\vec{b}\cdot\vec{b} - \frac{1}{2}\vec{a}\cdot\vec{b}$$

$$= \frac{1}{2}\left(|\vec{b}|^2 - |\vec{a}|^2\right).$$

Since the isoceles triangle has $|\vec{a}| = |\vec{b}|$, the vector product is zero, and $\overline{CD} \perp \overline{AB}$.

7. Let ABCD be a quadrilateral; i.e., A, B, C, D are distinct.

$\vec{M} = \frac{1}{2}(\vec{A} + \vec{B})$ $\qquad$ $\vec{N} = \frac{1}{2}(\vec{B} + \vec{C})$

$\vec{P} = \frac{1}{2}(\vec{C} + \vec{D})$ $\qquad$ $\vec{Q} = \frac{1}{2}(\vec{D} + \vec{A})$

M, N, P, Q are the midpoints of the sides.

We wish to show $\overline{MP}$ bisects $\overline{NQ}$.

Points of $\overline{MP}$: $x\vec{M} + (1 - x)\vec{P} \quad 0 \le x \le 1$

Points of $\overline{NQ}$: $y\vec{N} + (1 - y)\vec{Q}$

Intersection requires that

$$x\vec{M} + (1 - x)\vec{P} = y\vec{N} + (1 - y)\vec{Q}$$

$$x(\tfrac{1}{2}\vec{A} + \tfrac{1}{2}\vec{B}) + (1 - x)(\tfrac{1}{2}\vec{C} + \tfrac{1}{2}\vec{D}) = y(\tfrac{1}{2}\vec{B} + \tfrac{1}{2}\vec{C}) + (1 - y)(\tfrac{1}{2}\vec{D} + \tfrac{1}{2}\vec{A})$$

so $\frac{1}{2}x = \frac{1}{2}(1 - y)$ and $\frac{1}{2}(1 - x) = \frac{1}{2}y$

hence $x = y = \frac{1}{2}$.

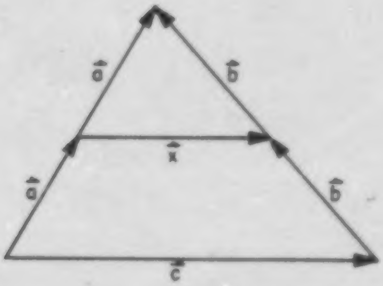
Thus $\overline{MP}$ intersects $\overline{NC}$ in a point which bisects both.

8. $\vec{x} = -\vec{a} + \vec{c} + \vec{b}$;

$\vec{x} = \vec{a} - \vec{b}$.

Adding, we have

$2\vec{x} = \vec{c}$, or $\vec{x} = \frac{1}{2}\vec{c}$.

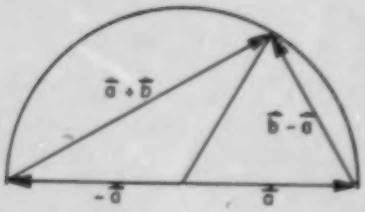


9. $(\overrightarrow{a} + \overrightarrow{b}) \cdot (\overrightarrow{b} - \overrightarrow{a})$

$= \overrightarrow{a} \cdot \overrightarrow{b} + \overrightarrow{b} \cdot \overrightarrow{b} - \overrightarrow{a} \cdot \overrightarrow{a} - \overrightarrow{a} \cdot \overrightarrow{b}$

$= |\overrightarrow{b}|^2 - |\overrightarrow{a}|^2$.

Since $|\overrightarrow{a}| = |\overrightarrow{b}|$, $|\overrightarrow{b}|^2 - |\overrightarrow{a}|^2 = 0$.



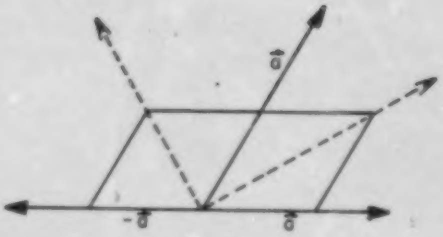
$\overrightarrow{a} + \overrightarrow{b}$
$\overrightarrow{b} - \overrightarrow{a}$
$-\overrightarrow{a}$
$\overrightarrow{a}$

10. As in Example 4, we use unit vectors to express the angle bisectors. Then, taking the vector product, we obtain

$$\frac{\overrightarrow{a}}{|\overrightarrow{a}|} + \frac{\overrightarrow{b}}{|\overrightarrow{b}|} \cdot \frac{\overrightarrow{b}}{|\overrightarrow{b}|} - \frac{\overrightarrow{a}}{|\overrightarrow{a}|}$$

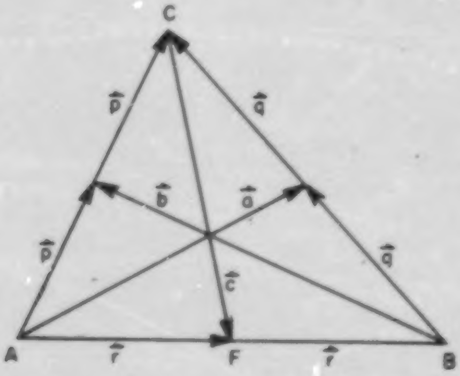
$$= \frac{\overrightarrow{a} \cdot \overrightarrow{b}}{|\overrightarrow{a}|\,|\overrightarrow{b}|} + \frac{\overrightarrow{b} \cdot \overrightarrow{b}}{|\overrightarrow{b}|^2} - \frac{\overrightarrow{a} \cdot \overrightarrow{a}}{|\overrightarrow{a}|^2} - \frac{\overrightarrow{a} \cdot \overrightarrow{b}}{|\overrightarrow{a}|\,|\overrightarrow{b}|}$$

$$= \frac{|\overrightarrow{b}|^2}{|\overrightarrow{b}|^2} - \frac{|\overrightarrow{a}|^2}{|\overrightarrow{a}|^2} = 0 .$$



$\overrightarrow{a}$
$-\overrightarrow{a}$
$\overrightarrow{a}$

11.



C
$\overrightarrow{p}$
$\overrightarrow{q}$
$\overrightarrow{b}$
$\overrightarrow{a}$
$\overrightarrow{p}$
$\overrightarrow{q}$
$\overrightarrow{c}$
A
$\overrightarrow{r}$
F
$\overrightarrow{r}$
B

$\overrightarrow{a} = 2\overrightarrow{r} + \overrightarrow{q}$ $\qquad$ $\overrightarrow{b} = -2\overrightarrow{r} + \overrightarrow{p}$ $\qquad$ $\overrightarrow{c} = -2\overrightarrow{p} + \overrightarrow{r}$

$\overrightarrow{a} = 2\overrightarrow{p} - \overrightarrow{q}$ $\qquad$ $\overrightarrow{b} = 2\overrightarrow{q} - \overrightarrow{p}$ $\qquad$ $\overrightarrow{c} = -2\overrightarrow{q} - \overrightarrow{r}$

$2\overrightarrow{a} = 2\overrightarrow{r} + 2\overrightarrow{p}$ $\qquad$ $2\overrightarrow{b} = 2\overrightarrow{q} - 2\overrightarrow{r}$ $\qquad$ $2\overrightarrow{c} = -2\overrightarrow{p} - 2\overrightarrow{q}$

$$\overrightarrow{a} + \overrightarrow{b} + \overrightarrow{c} = \overrightarrow{r} + \overrightarrow{p} + \overrightarrow{q} - \overrightarrow{r} - \overrightarrow{p} - \overrightarrow{q} = \overrightarrow{0} .$$

4-4. Proofs Using Polar Coordinates.

Polar coordinates are not particularly adapted for proving theorems of the type we have been discussing. The beauty and usefulness of this form will be more apparent in later chapters. Exercises using polar representation are, therefore, deferred. We have included two examples to illustrate the possibilities for polar coordinates at this point of our progress and to set the stage for the next section.

4-5. Choice of Method of Proof.

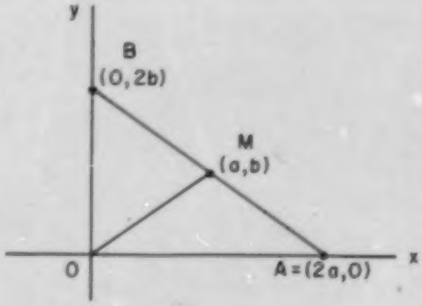
This section, which contains rather specific directions for problem solving, should be carefully read and discussed. Most of the Review Exercises which follow may be used to give the students experience in choosing and following through with some particular method. The solutions we present are merely the ones which occurred to us; they are not put forth as the only ones available or even the best of the many possibilities. As was said before, any mathematically sound presentation should be acceptable.

Review Exercises

1. $d(O,M) = \sqrt{a^2 + b^2}$.

$$d(A,M) = d(B,M) = \sqrt{(2a - a)^2 + b^2}$$

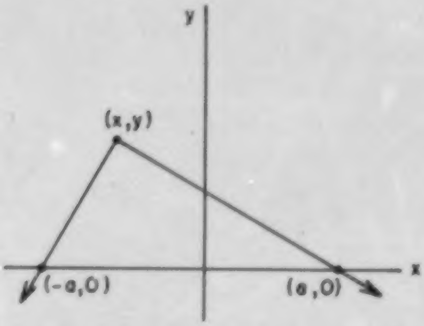
$$= \sqrt{a^2 + b^2} \ .$$



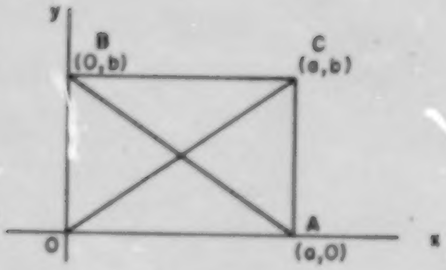
2. Let the fixed points be on the x-axis, as indicated in the figure. By multiplying the slopes of the sides of the angle we have

$$\frac{y}{x - a} \cdot \frac{y}{x + a} = -1 \ ,$$

$$y^2 = -x^2 + a^2 \ , \text{ or } \ x^2 + y^2 = a^2 \ .$$

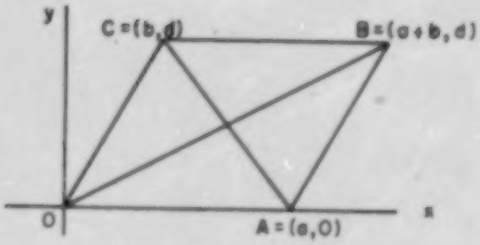


3.


y
B
(0,b)
C
(a,b)
A
O
(a,0)
x

$d(O,C) = \sqrt{a^2 + b^2}$ and $d(A,B) = \sqrt{a^2 + b^2}$

4.


y
C=(b,d)
B=(a+b,d)
O
A=(a,0)
x

The coordinates of B are $(a + b,d)$.

$$\big(d(O,A)\big)^2 + \big(d(A,B)\big)^2 + \big(d(B,C)\big)^2 + \big(d(C,O)\big)^2$$

$$= a^2 + (b^2 + d^2) + a^2 + (b^2 + d^2)$$

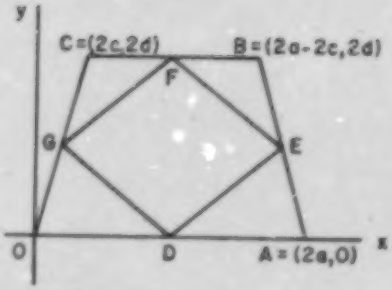
$$= 2(a^2 + b^2 + d^2)$$

$$\big(d(O,B)\big)^2 + \big(d(A,C)\big)^2 = \big((a + b)^2 + d^2\big) + \big((a - b)^2 + d^2\big)$$

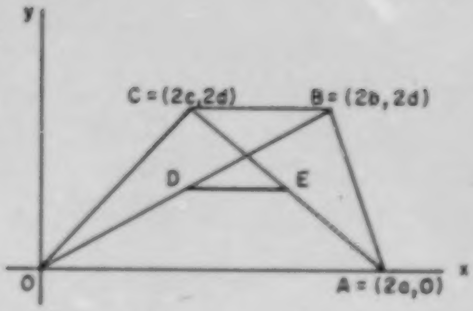
$$= 2(a^2 + b^2 + d^2).$$

5. $D = (a,0)$; $E = (2a - c,d)$;
$F = (a,2d)$; $G = (c,d)$.

From Exercise 10 of Exercises 4-2, we know that DEFG is a parallelogram; from Exercise 9 of Exercises 4-2, we know that DEFG is a rhombus if $\overline{DF} \perp \overline{GE}$. It is evident from the coordinates of the midpoints that $\overline{DF}$ is vertical and $\overline{GE}$ is horizontal.


y
C=(2c,2d)
B=(2a-2c,2d)
F
G
E
O
D
A=(2a,0)
x

6. D = (b,d) ; E = (a + c,d) .



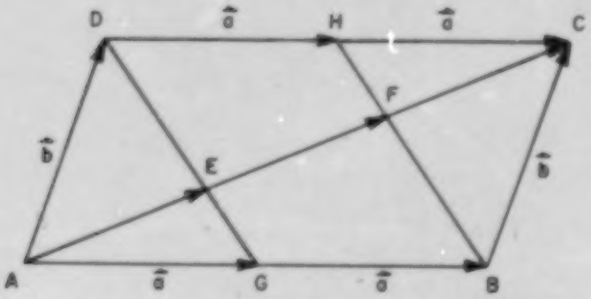
It is evident from the coordinates that $\overline{OA}$, $\overline{BC}$, and $\overline{DE}$ are horizontal and, hence, parallel.

d(O,A) - d(B,C) = 2a - (2b - 2c)

= 2(a - b + c).

d(D,E) = a + c - b.

7.



The vector from D to G is $\hat{a} - \hat{b}$; the vector from H to B is $2\hat{a} - \hat{b} - \hat{a} = \hat{a} - \hat{b}$; hence, $\overleftrightarrow{DG} \parallel \overleftrightarrow{HB}$. The vector from A to E may be represented by $x\hat{a} + (1 - x)\hat{b}$ or by $y(2\hat{a} + \hat{b})$. Setting these equal we have

$$x\hat{a} + (1 - x)\hat{b} = 2y\hat{a} + y\hat{b} .$$

Equating coefficients results in $x = 2y$, $y = 1 - x$. Solving these equations together gives us $y = \frac{1}{3}$. The vector from A to F may be represented by $x(2\hat{a}) + (1 - x)(\hat{a} + \hat{b})$ or by $y(2\hat{a} + \hat{b})$. Equating these, we obtain $y = \frac{2}{3}$.

8. Let D, E, and F be the midpoints of the sides, and let the perpendicular bisectors of $\overline{AB}$ and $\overline{BC}$ intersect at the origin. Since $\widehat{D}$ is perpendicular to the vector from A to B, $\frac{1}{2}(\widehat{A} + \widehat{B})\cdot(\widehat{B} - \widehat{A}) = 0$, or

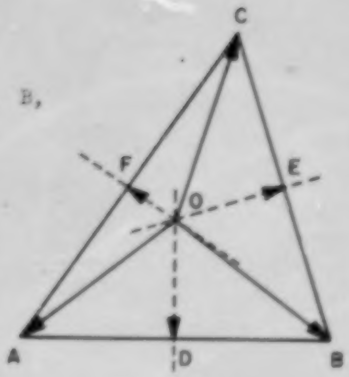
$\frac{1}{2}(\widehat{B}\cdot\widehat{B} - \widehat{A}\cdot\widehat{A}) = 0$; therefore

$|\widehat{B}|^2 = |\widehat{A}|^2$. Similarly, $|\widehat{A}|^2 = |\widehat{C}|^2$.

Since $\widehat{F} = \frac{1}{2}(\widehat{A} + \widehat{C})$, $\frac{1}{2}(\widehat{A} + \widehat{C})\cdot(\widehat{A} - \widehat{C})$

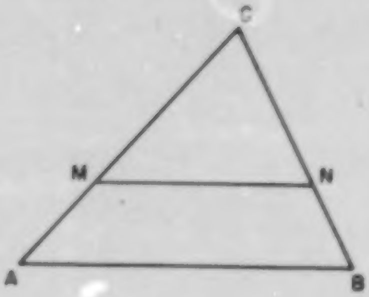
$$= \frac{1}{2}(\widehat{A}\cdot\widehat{A} - \widehat{C}\cdot\widehat{C})$$

$$= \frac{1}{2}(|\widehat{A}|^2 - |\widehat{C}|^2) .$$

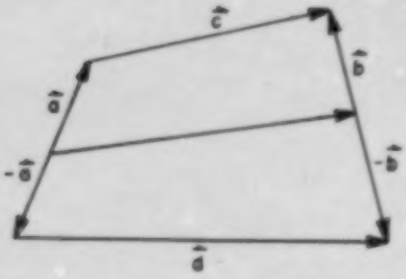


But since $|\widehat{A}|^2 = |\widehat{C}|^2$, $\frac{1}{2}(|\widehat{A}|^2 - |\widehat{C}|^2) = 0$, and $\widehat{F}$ is perpendicular to the vector from C to A. Consequently the perpendicular bisector of $\overline{AC}$ intersects the other two perpendicular bisectors at O.

9. Let M and N divide $\overline{AC}$ and $\overline{BC}$ in the same ratio, r . Then, $\widehat{M} - \widehat{N}$
$= (r\widehat{A} + (1 - r)\widehat{C}) - (r\widehat{B} + (1 - r)\widehat{C})$
$= r\widehat{A} - r\widehat{B} = r(\widehat{A} - \widehat{B})$.

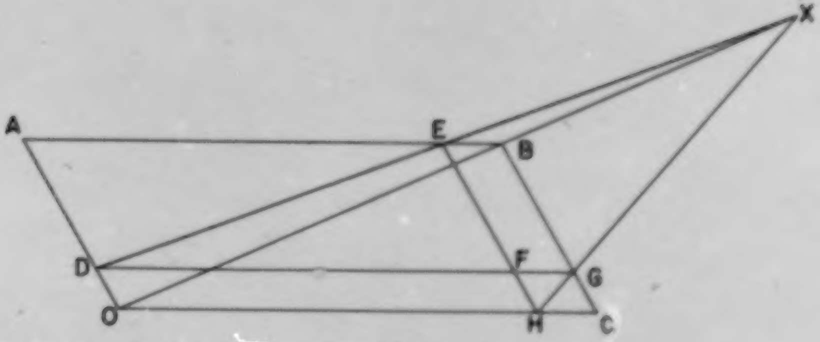


10.



$\widehat{x} = \widehat{a} + \widehat{c} - \widehat{b}$,
$\widehat{x} = -\widehat{a} + \widehat{d} -(-\widehat{b})$.
Adding, we obtain $2\widehat{x} = \widehat{c} + \widehat{d}$, or $\widehat{x} = \frac{1}{2}(\widehat{c} + \widehat{d})$.

11.



We are given parallelograms ABCO, AEFD, FGCH.

Define numbers d, h such that $\hat{D} = d\hat{A}$; $\hat{H} = h\hat{C}$.

We will express everything in terms of d, h, $\hat{A}$, $\hat{C}$ and assume all points are distinct.

The line through $\overline{DE}$ contains points $x\hat{D} + (1 - x)\hat{E}$

or $x(d\hat{A}) + (1 - x)(\hat{A} + h\hat{C})$.

The line through $\overline{HG}$ contains points $y\hat{H} + (1 - y)\hat{G}$

or $yh\hat{C} + (1 - y)(\hat{C} + d\hat{A})$.

For these two lines to intersect, we must have

$$(xd + 1 - x)\hat{A} + (1 - x)h\hat{C} = (1 - y)d\hat{A} + (yh + 1 - y)\hat{C} .$$

Thus we must have

$$yh + 1 - y = h - xh$$
$$xd + 1 - x = d - yd.$$

Solving this system we get, under condition that $h \neq 1 - d$,

$$y = \frac{d - 1}{h + d - 1} \qquad x = \frac{h - 1}{h + d - 1}$$

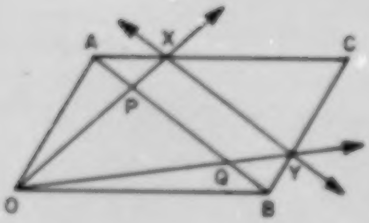
which puts the intersection at X such that

$$\hat{X} = \frac{hd}{h + d - 1}\hat{A} + \frac{hd}{h + d - 1}\hat{C} = \frac{hd}{h + d - 1}(\hat{A} + \hat{C}) .$$

From this we see immediately that X lies on the line containing $\overline{OB}$ since $\hat{A} + \hat{C} = \hat{B}$.

The restriction $h \neq 1 - d$ arises because in the case $h = 1 - d$, we get $\frac{|\hat{A}|}{|\hat{C}|} = h = 1 - d = \frac{1 - |\hat{D}|}{|\hat{A}|}$ which makes the parallelograms similar and the diagonals parallel.

12. Since $d(A,P) = d(Q,B)$, $\hat{P}$ can be represented by $\hat{A} + \hat{P}(\hat{B} - \hat{A})$ and $\hat{Q}$ by $\hat{B} + p(\hat{A} - \hat{B})$.
$\hat{x} = \hat{A} + k(\hat{C} - \hat{A})$ and $\hat{X} = q\hat{P}$,
so that
$\hat{A} + k(\hat{C} - \hat{A}) = q\hat{P}$,
$\hat{A} + k(\hat{A} + \hat{B} - \hat{A}) = q(\hat{A} + p(\hat{B} - \hat{A}))$,
$\hat{A} + k\hat{B} = q(1 - p)\hat{A} + qp\hat{B}$.



Equating coefficients, we have
$1 = q(1 - p)$ and $k = qp$;
therefore,

$k = \frac{p}{1 - p}$ and $\hat{X} = \hat{A} + \frac{p}{1 - p}\hat{B}$.

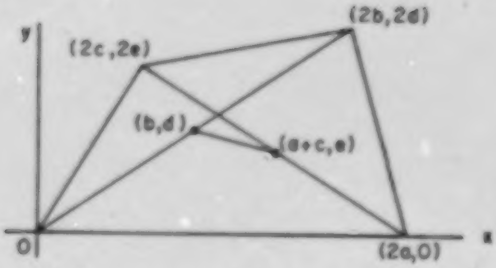
A similar argument gives us $\overline{Y} = \overline{B} + \frac{p}{1 - p}\hat{A}$.

Thus, $$\hat{X} - \hat{Y} = \hat{A} + \frac{p}{1 - p}\hat{B} - \hat{B} - \frac{p}{1 - p}\hat{A}$$

$$= (1 - \frac{p}{1 - p})(\hat{A} - \hat{B}) \;;$$

hence, $$\overleftrightarrow{XY} \,||\, \overleftrightarrow{AB} .$$

13. The sum of the squares of the lengths of the four sides is



$$(2a)^2 + (2b - 2a)^2 + (2d)^2 + (2b - 2c)^2 + (2d - 2e)^2 + (2c)^2 + (2e)^2$$

$$= 8a^2 + 8b^2 + 8c^2 + 8d^2 + 8e^2 - 8ab - 8bc - 8de .$$

The sum of the squares of the lengths of the diagonals is

$$(2b)^2 + (2d)^2 + (2c - 2a)^2 + (2e)^2$$

$$= 4a^2 + 4b^2 + 4c^2 + 4d^2 + 4e^2 - 8ac .$$

Subtracting these sums, we obtain,

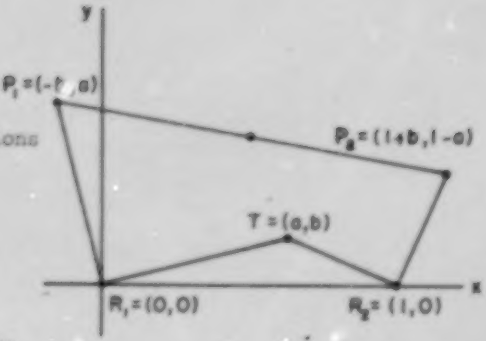
$$4a^2 + 4b^2 + 4c^2 + 4d^2 + 4e^2 + 8ac - 8ab - 8bc - 8de$$

$$= 4(a^2 + b^2 + c^2 + d^2 + e^2 + 2ac - 2ab - 2bc - 2de) .$$

The square of the length of the line segment joining the midpoints of the diagonals is

$$(a + c - b)^2 + (e - d)^2 = a^2 + b^2 + c^2 + d^2 + e^2 + 2ac - 2ab - 2bc - 2de .$$

14. We select coordinates for the two rocks and the tree as shown in the diagram. After marching the required distances and directions from the rocks, the positions P_1 and P_2 are located. The midpoint of $\overline{P_1P_2}$ is $(\frac{1}{2}, \frac{1}{2})$; therefore, the buried treasure is located at the center of the square whose side is determined by the two rocks. (The location of the tree is unimportant.)



Chapter 5

GRAPHS AND THEIR EQUATIONS

The material of this chapter starts with familiar content including much that has been encountered in earlier courses. The treatment is broader and deeper here than before. It is broader because we now have analytic representations in rectangular, polar, vector, and parametric forms. It is deeper because we take account of some troublesome details and special cases that are not adequately treated on a more elementary level. The work is consequently a bit more difficult, but also more rewarding.

We call particular attention to the treatment of related polar equations, and of paths, as distinguished from curves. Neither treatment is met in a traditional first course in analytic geometry, but we feel that they illuminate some significant mathematical content that is appropriate to this work.

There are many exercises, but, as has been mentioned before in this book, they need not all be assigned. We particularly urge the teacher to exploit a viewpoint we recommended to students. Stress the dynamic aspect of the relationship between geometry and algebra. Some appropriate questions here are, "What would be the effect in the graph if we changed this 5 to -5?"; "What change would we have to make in the equation if we wanted to raise the graph 3 units?; if we wanted a larger circle?; if we wanted only the portion in the first quadrant?"; "What kind of graphs would we get if we replaced this 6 by a variable m , and then took larger and larger values of m ?".

Exercises 5-2

1. $y = 3$
2. $x = -5$
3. $y = x$ and $y = -x$; or $x^2 = y^2$
4. $y = \pm 2x$; or $y^2 = 4x^2$
5. $r = a$; or $x^2 + y^2 = a^2$
6. $(x - 3)^2 + (y + 2)^2 = a^2$

7. $x = -1$

8. $3x - 7y - 14 = 0$

9. $\sqrt{5}|x + y - 2| = \sqrt{2}|x + 2y + 2|$; or

$(\sqrt{5} + \sqrt{2})x + (\sqrt{5} + 2\sqrt{2})y - 2\sqrt{5} + 2\sqrt{2} = 0$, and

$(\sqrt{5} - \sqrt{2})x + (\sqrt{5} - 2\sqrt{2})y - 2\sqrt{5} - 2\sqrt{2} = 0$.

10. $y^2 = 8x$

11. If $P = (x,y)$ is a point of the locus, then the distance from P to the line is $\frac{|2x + y + 2|}{\sqrt{5}}$, and from P to the point $(2,0)$ is $\sqrt{(x - 2)^2 + (y + 1)^2}$. The statement of equality of these two distances yields our equation: $x^2 - 4xy + 4y^2 - 28x + 6y + 21 = 0$.

12. $9x^2 + 25y^2 = 225$

13. $7x^2 - 9y^2 = 63$

14. $18x^2 + 48xy + 7y^2 - 156x - 68y + 142 = 0$

15. $5x - 6y + 17 = 0$

16. $\left((x - x_1)^2 + (y - y_1)^2\right)\left((x - x_2)^2 + (y - y_2)^2\right) = k^2$, $k > 0$

17. $-3 \leq y \leq 3$

18. $x^2 + y^2 \geq 25$

19. $-1 \leq x \leq 1$

20. $(x - 1)^2 + (y - 3)^2 \leq 2^2$, or $x^2 + y^2 - 2x - 6y + 6 \leq 0$

21. $y > \frac{5}{2}$

22. $x^2 + 8y \geq 16$

23. $y^2 \leq 100 - 20x$

24. $-6 < x < 6$; or $|x| < 6$

25. $x^2 + y^2 < (8.08)^2$; or $x^2 + y^2 < 65.2864$

5-3. Parametric Representation.

The content and treatment of the material in this section are closely related to the physical and scientific applications that pupils will meet in other classes and in later work. Science teachers in the school should be shown this section, and their cooperation solicited in devising laboratory experiments along the lines suggested.

Exercises 5-3

1.

t	0	1	2	3	4	5	6	7	8	9	10
x	0	2	8	18	32	50	72	98	128	162	200
y	0	3	12	27	48	75	108	147	192	243	300

2.

t	0	1	2	3	4	5	6	7	8	9	10
x	0	176	352	528	704	880	1056	1232	1408	1584	1760
y	0	16	64	144	256	400	576	784	1024	1296	1600

3. $\begin{cases} x = 5t , \\ y = 2 . \end{cases}$

4. $\begin{cases} x = -6 , \\ y = 2t . \end{cases}$

5. $\begin{cases} x = .3t , \\ y = .4t . \end{cases}$

6. $\begin{cases} x = -6 + \frac{7}{5}t , \\ y = 1 + \frac{24}{5}t . \end{cases}$

7. Eliminating the parameter gives $y = x^2$. With the usual placement of the axes this means that the point starts from rest at the origin and moves steadily to the right as it moves more and more rapidly upward. Its path is along a parabola whose vertex is at the origin and which is concave upward. Since we assume $t > 0$, the point travels on only the right half of the parabola. 25.9 units.

8. For the line $4x - 3y + 2 = 0$ we have direction numbers for the normal, $(4,-3)$. Therefore we may take direction numbers for the line as either $(3,4)$, or $(-3,-4)$. Since no sense of direction along the line is specified we must consider both. If we use direction consines then the displacement along the line will be one unit for each unit interval of the parameter t. Since the given rate is 10 units per second we must now take direction numbers ten times the direction cosines, i.e., $\left(10(\frac{3}{5}), 10(\frac{4}{5})\right)$. Since the point goes through $(1,2)$ at the time when $t = 3$, the elapsed time after that is indicated by $t - 3$. We have, in the first case, therefore,

$$\begin{cases} x = 1 + 6(t - 3), \\ y = 2 + 8(t - 3); \end{cases} \quad \text{or} \quad \begin{cases} x = -17 + 6t, \\ y = -22 + 8t; \end{cases}$$

and in the second case,

$$\begin{cases} x = 19 - 6t, \\ y = 26 - 8t. \end{cases}$$

In the first case, when $t = 0$ the position is $(-17,-22)$, and when $t = 10$ the position is $(43,58)$. In the second case, when $t = 0$ the position is $(19,26)$, and when $t = 10$ the position is $(-41,-54)$.

9. Refer to the solution of (8) above.

$$\begin{cases} x = 3 + \dfrac{15}{\sqrt{13}}t, \\ y = 0 - \dfrac{10}{\sqrt{13}}t; \end{cases} \quad \text{or} \quad \begin{cases} x = 3 - \dfrac{15}{\sqrt{13}}t, \\ y = 0 + \dfrac{10}{\sqrt{13}}t. \end{cases}$$

10. Assume $t_1 > t_0$. Direction numbers for the line are $(c - a, d - b)$, and direction cosines $\dfrac{c - a}{\sqrt{(c - a)^2 + (d - b)^2}}, \dfrac{d - b}{\sqrt{(c - a)^2 + (d - b)^2}}$.

The velocity of the point along the line is $\dfrac{\sqrt{(c - a)^2 + (d - b)^2}}{t_1 - t_0}$,

and this is the factor by which we must multiply the direction cosines so that unit intervals of the parameter t correspond properly to displacements along the line. Since the point goes through (a,b) at time t_0 we indicate with our parameter t the elapsed time since then, $t - t_0$. Therefore we have the parametric equations:

$$\begin{cases} x = a + \dfrac{\sqrt{(c-a)^2 + (d-b)^2}}{t_1 - t_0} \dfrac{c - a}{\sqrt{(c-a)^2 + (d-b)^2}}(t - t_0) , \\ y = b + \dfrac{\sqrt{(c-a)^2 + (d-b)^2}}{t_1 - t_0} \dfrac{d - b}{\sqrt{(c-a)^2 + (d-b)^2}}(t - t_0) . \end{cases}$$

These formidable equations become:

$$\begin{cases} x = a + \dfrac{c - a}{t_1 - t_0}(t - t_0) , \\ y = b + \dfrac{d - b}{t_1 - t_0}(t - t_0) . \end{cases}$$

You may easily verify from these equations that when $t = t_0$ the position is (a,b), and when $t = t_1$ the position is (c,d) .

11. Assume t in seconds. The point moves from the point $(1,0)$ to the point $(-1,0)$ and back again, making a round trip in 2π seconds. It starts from rest at $(1,0)$, increases its speed until it reaches the origin, then slows down until it comes to rest momentarily at $(-1,0)$, then reverses the process endlessly. Its maximum speed occurs each time at the origin. (By methods of the calculus this maximum speed can be shown to be one unit per second at that instant.) Such motion is called a "simple harmonic motion" and has many physical applications.

t	0	1	2	3	4	5	6	7	8	9	10
x	1	.540	-.418	-.990	-.652	.287	.961	.752	-.150	-.913	-.836

At the end of one minute $t = 60$, and Table II does not give corresponding values for $\cos t$. We use the fact that $\cos t$ is periodic, of period 2π . (These matters will be developed further in the next chapter.)

We express 60 as a multiple of π and a remainder less than π, which we find by dividing 60 by a suitable decimal equivalent of π . Tables I and II are given correct to three significant figures and a careless student may then take 3.14 as a proper equivalent of π . However, any inaccuracy in this approximation will be multiplied by a factor of about 20 and will give us a seriously inaccurate answer.

It is not our intention to enter into an extended discussion of significant figures and accuracy of computation, but in this exercise we caution that we must choose an appropriate approximation of π .

We assume $t = 60 = 60.0000$, and use $\pi \approx 3.1416$ and obtain $60.0000 = 19\pi + .3096$, which we write briefly as $60 = 19\pi + .310$. Therefore $\cos 60 = \cos(19\pi + .310) = - \cos .310 = - .952$.

In the same way we assume t for one hour to equal 3600.0000000, not 3600, and then take the proper approximation, $\pi \approx 3.141593$. Then $3600,0000000 = 1145\pi + 2.876015$, or $3600.0000000 = 1146\pi - .285578$, which we write more briefly as $3600 = 1146\pi - .286$. Thus $\cos 3600 = \cos(1146\pi - .286) = \cos(-.286) = \cos .286 \approx .959$.

You need not belabor the details of approximate computation, but this is a good place to show the need for a proper approximation for π. It is also a good place to show that when we are working with measurements and we add zeros to the dividend in division we are assuming more and more accuracy in its determination. A measurement of 10. inches is less accurate than one of 10.0 inches which is in turn less accurate than a measurement of 10.00 inches. We particularly warn against the error of dividing a 10 inch length into three equal parts and writing the length of one part as 3.3333.... inches!

12. The motion could be that of an object dropped from an altitude of 500 feet, in which case we assume no air resistance, and a value of 16 feet per second per second as the acceleration due to gravity. A value of y represents the altitude, in feet, above the surface of the earth, at corresponding time t, in seconds after the instant of release. The change of sign of y in the interval $t = 5$ to $t = 6$ can be interpreted to mean that the object reaches the surface of the earth in that interval. The negative values of y afterwards would indicate the depth below the surface, if the fall continued down a vertical shaft.

t	0	1	2	3	4	5	6	7	8	9	10
y	500	484	436	356	244	100	-76	-284	-524	-796	-1100

13. (Refer to the solution of Exercise 12) This equation could represent the motion of an object hurled upward at 64 feet per second from an altitude of 120 feet.

t	0	1	2	3	4	5	6	7	8	9	10
y	120	168	184	168	120	40	-72	-216	-392	-600	-840

14. (Refer to the solution of Exercise 1F.) This equation could describe a simple harmonic motion with these conditions: The point starts from a position of rest at the origin; moves, in the next $\frac{\pi}{4}$ seconds, to its farthest right position at (4,0) where it halts momentarily and reverses direction to move to its farthest left position at (-4,0), arriving there in an additional $\frac{\pi}{2}$ seconds. It accelerates from (4,0) to the origin where it attains its maximum velocity, then decelerates from the origin to (-4,0), and so on making a round trip in π seconds. Such equations of motions occur in the study of vibrations, and of variations of an alternating current.

t	0	1	2	3	4	5	6	7	8	9	10
x	0	3.636	-3.032	-1.104	3.956	-2.192	-2.124	3.864	-1.184	-2.980	3.668

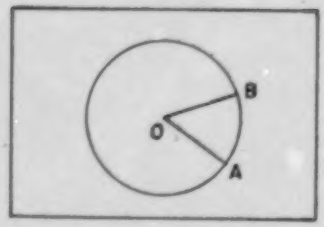
15. (Refer to the solution of Exercise 11.) The point now starts from (1,0) and moves to (3,0) and back, as before, making the round trip in 2π seconds.

16. We must assume they start at the same instant, in which case the variable t has the same interpretation in both equations. Therefore $\cos t = 2 - \cos t$, from which we get $\cos t = 1$ and $t = 0, 2\pi, 4\pi, \ldots$. For these values of t, $x = 1$, therefore the points start together at (1,0), and rendezvous there every 2π seconds thereafter.

5-4. Parametric Equations of the Circle and the Ellipse.

Focal phenomena are familiar enough to physics, but it is interesting to see how the associated mathematical analysis can be used in other situations. Authors in recent publications have applied these concepts in such areas as: epidemiology, to study the spread and control of disease; demography, to study the distributions of groups of people; bacteriology, to study the spread or control of bacterial growth; communication theory, to study the distribution of "information", and so on. We leave these for later years, and concern ourselves now with the simplest and most natural of the applications of parametric equations of the circle, that is, circular paths.

The teacher is urged to make a simple visual aid: The essential features are two movable radii $\overline{OA}$ and $\overline{OB}$ mounted on a panel of suitable size. Two students can then give independent motions to points on the rim of the circle. This model will be particularly useful when you get to problems of "meeting" or "overtaking".



Exercises 5-4

1. $\begin{cases} x = 10 \cos \theta , \\ y = 10 \sin \theta . \end{cases}$

2. We assume t in seconds. A clockwise rotation means that as t increases from 0 , θ decreases from 0 , and in this case a rate of 4 rps gives the angular displacement, $-8\pi t$. The equations are

$$\begin{cases} x = 10 \cos(-8\pi t) , \\ y = 10 \sin(-8\pi t) . \end{cases}$$

3. Consider $x = a \cos(b + \omega t)$. Since the radius is 6 inches, then $a = 6$ and we are committed to inches as the measure of x .

Since the numbers 0 and 60 are assigned to the 12 o'clock position the units of rotation in this problem are intended to be minutes. The angular position of any point on the rim can be given in terms of these m-units, measured from the 12 o'clock position, or in terms of the usual θ , in radian units from the polar axis. Thus the 2 o'clock position can be described by $m = 10$, and also by $\theta = \frac{\pi}{6}$. Since we rotate clockwise at the rate of one rotation in 60 minutes we have ω , the directed rate of angular displacement, equal to 1 m-unit per minute, or $\frac{-\pi}{30}$ radians per minutes.

If in the equation $x = a \cos(b + \omega t)$ we use radian units for b we have $b = \frac{\pi}{2}$, since we start from the 12 o'clock position. Finally, since we are asked for the path during one hour, we take $0 < t < 60$. The result of all this discussion is the following pair of equations:

$$\begin{cases} x = 6 \cos(\frac{\pi}{2} - \frac{\pi}{30}t) , \\ y = 6 \sin(\frac{\pi}{2} - \frac{\pi}{30}t) , \end{cases} \quad 0 \leq t < 0 .$$

t is the time in minutes, x and y are in inches, and the angle is measured as usual in radians, counterclockwise from the polar axis.

4. $$\begin{cases} x = 4 + 3\cos\theta , \\ y = 3\sin\theta . \end{cases}$$

5. $$\begin{cases} x = 4\cos\theta , \\ y = 6 + 4\sin\theta . \end{cases}$$

6. $$\begin{cases} x = 4 + 3\cos(-\frac{\pi}{2} - 4\pi t), \\ y = 3\sin(-\frac{\pi}{2} - 4\pi t) . \end{cases}$$

Note: These equations supply information about the starting position $(-\frac{\pi}{2})$, and the direction and speed of rotation (-4π), but for purposes of computation they may be replaced by the equivalent equations,

$$\begin{cases} x = 4 + \cos(\frac{\pi}{2} + 4\pi t) , \\ y = -3\sin(\frac{\pi}{2} + 45t) . \end{cases}$$

These latter equations show that the path of the point P of exercise 6 is the reflection in the x-axis of the path of the point P' whose equations are

$$\begin{cases} x' = 4 + \cos(\frac{\pi}{2} + 4\pi t) , \\ y' = 3\sin(\frac{\pi}{2} + 4\pi t) . \end{cases}$$

The point P' starts at the highest point of its path and moves counterclockwise, as we should expect the reflected point to do.

7. $$\begin{cases} x = 4\cos(\frac{\pi}{2} + 6\pi t) , \\ y = 6 + 4\sin(\frac{\pi}{2} + 6\pi t) . \end{cases}$$

8. The point moves around a circle whose center is the origin and whose radius is 4. The point starts from the 3 o'clock position and moves counterclockwise at the rate of $\frac{1}{2}$ rotation per second.

9. The point moves around a circle whose center is the origin and whose radius is 6. It starts from the 12 o'clock position and moves clockwise at the rate of $\frac{1}{2}$ rps.

Note: In Solutions 10-16 the paths are all circular, and we shall condense the information which could be written out in full as in (8) and (9) above.

10. Circle; center, origin; $r = 8$; start, 9 o'clock position; direction, clockwise; rate, $\frac{3}{2}$ rps.

11. Circle; center, origin; $r = 10$; start, 6 o'clock position; direction, counterclockwise; rate, 5 rps.

12. Circle; center, $(4,0)$; radius, 1 ; start, 3 o'clock position; direction, counterclockwise; rate, 3 rps.

13. Circle; center, $(0,-3)$; radius, 1 ; start, 3 o'clock position; direction, counterclockwise; rate, 4 rps.

14. Circle; center, $(2,5)$; radius, 1 ; start, 3 o'clock position; direction, counterclockwise; rate, 6 rps.

15. Circle; center, (a,c) ; radius, b ; start, 3 o'clock position; direction, counterclockwise; rate, 1 rps.

16. Circle; center, (p,r) ; radius q ; start, at the angular position $-\alpha$ on the circle; direction, counterclockwise if $n < 0$, no motion at all if $n = 0$; rate, n rps.

17. (a) Circle; center, origin; radius, 6 ; start, 3 o'clock position; direction, counterclockwise; rate, 2 rps.

(b)

t	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
x	6.00	1.84	-4.88	-4.88	1.84	6.00	1.84	-4.88	-4.88	1.84	6.00
y	0	5.71	3.49	-3.49	-5.71	0	5.71	3.49	-3.49	-5.71	0

(c)
$$\begin{cases} x = 6\cos(\frac{\pi}{2} + 4\pi t) , \\ y = 6\sin(\frac{\pi}{2} + 4\pi t) . \end{cases}$$

(d)
$$\begin{cases} x = 6\cos(-2\pi t) , \\ y = 6\sin(-2\pi t) . \end{cases}$$

(e) Since the first and third points move in opposite directions, they will meet when the sum of their angular displacements equals their original separation, and, after that, when their additional angular displacements add to an integral multiple of 2π . That is, $2\pi t + 4\pi t = 0$, since they start together, from which $t = 0$, and the points are at $(6,0)$. After that, $2\pi t + 4\pi t = 2\pi , 4\pi , 6\pi ,$ $\ldots$, that is, $t = \frac{1}{3} , \frac{2}{3} , 1 , \frac{4}{3} , \ldots$. The points start together, and meet every $\frac{1}{3}$ second thereafter. The corresponding points are $(6,0)$, $(-3,-5.196)$, $(-3,5.196)$, $(6,0)$, $(-3,-5.196)$, $\ldots$.

(f) As in the previous part, we add the angular displacements, and find the first meeting point when this sum is equal to their original angular separation: that is, when $2\pi t + 4\pi t = \frac{\pi}{2}$. Thus they meet first when $t = \frac{1}{12}$, at the point $(5.196, -3)$. Then we find, as above, their subsequent meetings take place every $\frac{1}{3}$ second, which should be expected, since the first and second points are traveling at the same rate. The meetings therefore take place when $t = \frac{1}{12}, \frac{5}{12}, \frac{9}{12}, \frac{13}{12}, \ldots$, at $(5.196, -3)$, $(-5.196, -3)$, $(6, 0)$, $(5.196, -3)$, ...

18. (a)

$$A: \begin{cases} x = \frac{1}{2\pi}\cos(\frac{7}{6}\pi - \frac{2}{3}\pi t), \\ y = \frac{1}{2\pi}\sin(\frac{7}{6}\pi - \frac{2}{3}\pi t). \end{cases}$$

$$B: \begin{cases} x = \frac{1}{2\pi}\cos(\frac{11}{6}\pi - \frac{1}{2}\pi t), \\ y = \frac{1}{2\pi}\sin(\frac{11}{6}\pi - \frac{1}{2}\pi t). \end{cases}$$

$$C: \begin{cases} x = \frac{1}{2\pi}\cos(\frac{1}{2}\pi + \frac{2}{5}\pi t), \\ y = \frac{1}{2\pi}\sin(\frac{1}{2}\pi + \frac{2}{5}\pi t). \end{cases}$$

(b) A: When $t = 0, 3, 6, 9$, position is $(-\frac{\sqrt{3}}{4\pi}, -\frac{1}{4\pi})$;

When $t = 1, 4, 7, 10$, position is $(0, \frac{1}{2\pi})$;

When $t = 2, 5, 8$, position is $(\frac{\sqrt{3}}{4\pi}, -\frac{1}{4\pi})$.

B: When $t = 0, 4, 8$, position is $(\frac{\sqrt{3}}{4\pi}, -\frac{1}{4\pi})$;

When $t = 1, 5, 9$, position is $(-\frac{1}{4\pi}, -\frac{\sqrt{3}}{4\pi})$;

When $t = 2, 6, 10$, position is $(-\frac{\sqrt{3}}{4\pi}, \frac{1}{4\pi})$;

When $t = 3, 7$, position is $(\frac{1}{4\pi}, \frac{\sqrt{3}}{4\pi})$.

C: When $t = 0, 5, 10$, position is $(0, .159)$;
When $t = 1, 6$, position is $(-.151, .049)$;
When $t = 2, 7$, position is $(-.094, -.129)$;
When $t = 3, 8$, position is $(.094, -.129)$;
When $t = 4, 9$, position is $(.151, .049)$;

(c) By the methods of the solution of Exercise 17 we find:

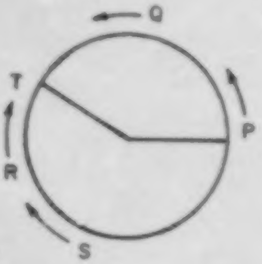
(1) A and C meet when $t = .625$, at $(-.112, .112)$;
(2) B and C meet when $t = 1.480$, at $(-.152, -.046)$;
(3) A and C meet when $t = 2.500$, at $(0, -.159)$;
(4) B and C meet when $t = 3.700$, at $(.159, -.008)$;
(5) A and C meet when $t = 4.375$, at $(.112, .112)$.

(d) By the methods already referred to we find that A and C meet in $\frac{5}{8}$ seconds and every $\frac{15}{8}$ seconds thereafter. That is, their meetings take place at times $t = \frac{5}{8} + \frac{15}{8}p$, where p is a positive integer. In the same way, we find that B and C meet in $\frac{40}{27}$ seconds and every $\frac{20}{9}$ seconds thereafter. That is, the B and C meetings take place when $t = \frac{40}{27} + \frac{20}{9}q$, where q is a positive integer. If A, B, and C are all to meet, there must be a time at which the A, C, and the B, C meetings occur simultaneously. That is, there must be positive integral values of p and q such that $\frac{5}{8} + \frac{15}{8}p = \frac{40}{27} + \frac{20}{9}q$. This equation is equivalent to $81p - 96q = 37$. In this equation, however, the left member is evenly divisible by 3 but the right member is not, therefore there can be no integral values of p and q to satisfy it. Therefore there can be no common meeting of A, B, and C.

19. Since the points move in reflected paths with respect to the y-axis, the second point must start from the position symmetric to A, that is, at $(-\pi, 0)$, where the angular displacement from A is π. Therefore the equations for the second point are

$$\begin{cases} x = r\cos(\pi - 4\pi t), \\ y = r\sin(\pi - 4\pi t). \end{cases}$$

20. (a) Assume a unit circle, time in seconds, and angular velocity in radians per second. The 10 o'clock position, T, has an angular displacement of $\frac{5\pi}{6}$. Since point P arrives at position T in 10 seconds, its angular velocity is $\frac{5\pi}{60}$ or $\frac{\pi}{12}$. In the same way the angular velocities of Q, R, and S are $\frac{\pi}{30}$, $-\frac{\pi}{60}$, and $-\frac{2\pi}{30}$ or $-\frac{\pi}{15}$.



Therefore, as before, the equations of motion are:

$$P: \begin{cases} x = \cos \frac{\pi}{12} t \,, \\ y = \sin \frac{\pi}{12} t \,. \end{cases}$$

$$Q: \begin{cases} x = \cos(\frac{\pi}{2} + \frac{\pi}{30} t \,) \,, \\ y = \sin(\frac{\pi}{2} + \frac{\pi}{30} t \,) \,. \end{cases}$$

$$R: \begin{cases} x = \cos(\pi - \frac{\pi}{60} t \,) \,, \\ y = \sin(\pi - \frac{\pi}{60} t \,) \,. \end{cases}$$

$$S: \begin{cases} x = \cos(\frac{3\pi}{2} - \frac{\pi}{15} t \,) \,, \\ y = \sin(\frac{3\pi}{2} - \frac{\pi}{15} t \,) \,. \end{cases}$$

(b) By the methods of the solution of the previous exercise we find that the meetings of the following pairs take place at the indicated times (where a , b , c , d , are positive integers):

Q and R , when $t_1 = 10 + 40a$;

Q and S , when $t_2 = 10 + 20b$;

P and R , when $t_3 = 10 + 20c$;

P and S , when $t_4 = 10 + \frac{40}{3}d$.

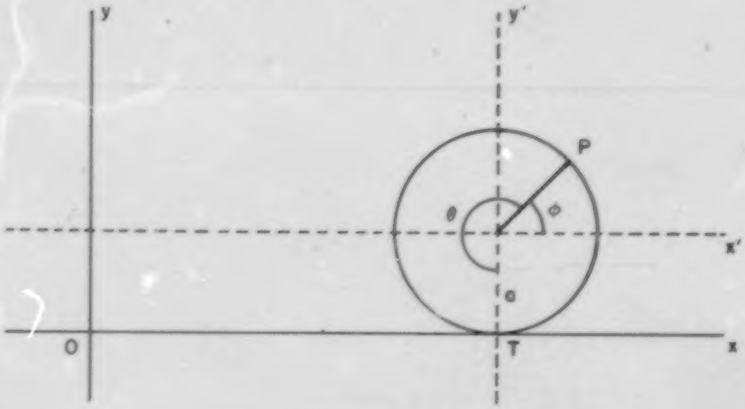
We verify that when a , b , c , d are all zero, the values of t_1 , t_2 , t_3 , t_4 , are all equal to 10 , as required by the statement of the problem. If there is to be a simultaneous meeting at another time, there must be values of a , b , c , d other than zero for which these times are equal. Clearly, if we take $d = 3$ or any multiple of 3, we can find such values. When $d = 3$, then $t_4 = 10 + 40 = 50$. Successive multiples of 3 as values of d give values of t_4 : 10 , 50 , 90 , 130 , ... , and these are clearly possible values of t_1 , t_2 , and t_3 , also. That is, the simultaneous meetings take place every 40 seconds after the first such meeting. The angular positions of these meetings are found to be $\frac{5\pi}{6}$, $\frac{25\pi}{6}$, $\frac{15\pi}{2}$, $\frac{130\pi}{12}$, ...

Questions of meeting or overtaking on circular paths are related to important problems in space exploration. Consider the complications that arise: the paths in space are not circular but essentially elliptical; the paths are not along the same ellipse, and the different ellipses are not usually in the same plane, so that we must not consider the meeting points (they would be catastrophic), but the points of nearest approach; the velocities along these paths are not uniform but variable in very complicated ways. The solutions to the exercises in our text are essential first steps in arriving at the level of ability needed to solve the difficult problems of astrogation that arise in space travel.

5-5. Parametric Equations of the Cycloid.

The physical applications of the cycloid are interesting indeed but their analysis is beyond the scope of this book. Students who are interested in photography can make photographs of a cycloid by taking a time exposure of a flashlight attached to an automobile wheel as it rolls along the road.

We give another derivation of the equations of the cycloid which uses the idea of a transformation of coordinates. You may wish to leave this derivation until you have reached the more complete treatment of transformation in Chapter 10.



Since $d(O,T)$ = length of $\widehat{PT} = a\theta$, the coordinates of the center of the circle are $(a\theta,a)$. We take this point as origin of an x'- , y'-coordinate system, hence $P = (x,y)$ becomes

$$P = (x',y') ,$$

where

$$\begin{cases} x = x' + a\theta , \\ y = y' + a . \end{cases}$$

But in this new coordinate system

$$\begin{cases} x' = a \cos \phi , \\ y' = a \sin \phi . \end{cases}$$

Since $\phi = \frac{3\pi}{2} - \theta$ we have

$$\cos \phi = -\sin \theta \quad \text{and} \quad \sin \phi = -\cos \theta ,$$

therefore

$$\begin{cases} x' = -a \sin \theta , \\ y' = -a \cos \theta , \end{cases}$$

Therefore, finally

$$\begin{cases} x = -a \sin \theta + a\theta , \\ y = -a \cos \theta + a \ ; \end{cases} \quad \text{or} \quad \begin{cases} x = a(\theta - \sin \theta) , \\ y = a(1 - \cos \theta) . \end{cases}$$

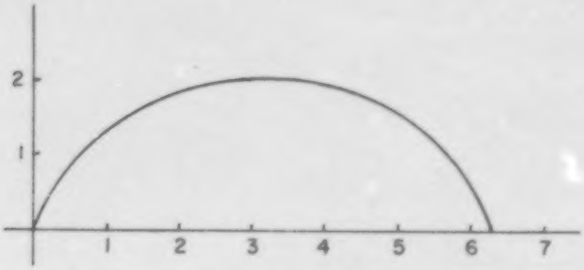
Exercises 5-5

1. $\begin{cases} x = \theta - \sin \theta , \\ y = 1 - \cos \theta . \end{cases}$

The intervals suggested indicate degree measure, but it would be an error to use these measures in the equations above, since the equations were derived on the basis of radian measure for θ . We may revise the formulas to suit degree measure, or convert the intervals to radian measure. The latter procedure is the easier and the one we follow.

θ degrees	0	30	60	90	120	150	180	210	240	270	300	330	360
θ radians	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
x	0	0	.2	.6	1.2	2.1	3.1	4.2	5.1	5.7	6.1	6.3	6.3
y	0	.1	.5	1.0	1.5	1.9	2.0	1.9	1.5	1.0	.5	.1	0

The values of x and y are computed to the nearest tenth, and the graph is sketched below.



2. The height of the rectangle is the diameter of the generating circle whose radius is therefore equal to 3. The base of the rectangle is as long as the circumference of that circle and is therefore 6π. The equations of the cycloid are

$$\begin{cases} x = 3(\phi - \sin \phi) , \\ y = 3(1 - \cos \phi) . \end{cases}$$

3. We have $a = 3$ inches, and equations for the graph,

$$\begin{cases} x = 3(\phi - \sin \phi) , \\ y = 3(1 - \cos \phi) . \end{cases}$$

The angular velocity is given as 4 rps which means that $\omega = 8\pi$ radians per second. Since $\theta = \omega t$ the equations above become

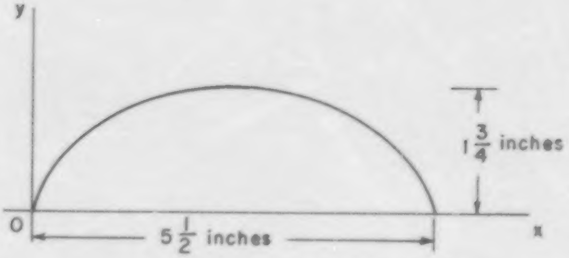
$$\begin{cases} x = 3(8\pi t - \sin 8\pi t) , \\ y = 3(1 - \cos 8\pi t) . \end{cases}$$

t	.1	.2	.3	.4	.5
x	5.77	17.93	19.75	28.38	37.68
y	5.42	2.08	2.08	5.42	0

To compute these values we had to find functions of angles whose radian measures exceeded 1.60, which is as far as our Table II goes. We must use the procedure explained in the solution to Exercise 5-3, Number 11. Thus $\sin .8\pi = \sin 2.51 = \sin(\pi - 2.51) = \sin .63 = .589$, and so on.

P will reach its first high point at the end of the first half turn which will occur at the end of the first $\frac{1}{8}$ second. When $t = .125$, $P = (9.4 , 6)$.

4. (a) All cycloids have the same shape, therefore an accurate scale drawing requires any carefully drawn cycloid and a properly chosen scale. The width of one arch is $2\pi a$, and the height is $2a$, where a is the radius of the generating circle. In this case the base line represents 66 inches, or $2\pi a$. Therefore $a = 10\frac{1}{2}$ inches. We suggest a scale of 1:12 which means that the drawing should be $5\frac{1}{2}$ inches across and $1\frac{3}{4}$ inches high.



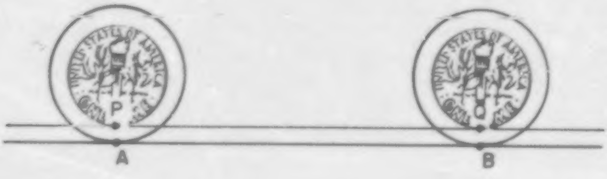
(b) We have

$$\begin{cases} x = a(\phi - \sin \phi) , \\ y = a(1 - \cos \phi) ; \end{cases} \qquad a = 10\frac{1}{2} .$$

We must correct the linear rate of 30 mph into an angular rate of rotation for a wheel with 66 inch circumference. A rate of 30 mph $= \frac{30 \cdot 5280 \cdot 12}{60}$ inches per minute $= \frac{6 \cdot 5280}{66}$ rotations per minute $= \frac{5280}{11} 2\pi$ radians per minute. Therefore $\omega = \frac{10560}{11} \pi$ and $\theta = \frac{10560}{11} \pi t$. Finally we have the equations of motion with values for x and y in inches, and t in minutes:

$$\begin{cases} x = \frac{21}{2}\left(\frac{10560}{11} \pi t - \sin \frac{10560}{11} \pi t\right) , \\ y = \frac{21}{2}\left(1 - \cos \frac{10560}{11} \pi t\right) . \end{cases}$$

You may wish to present the following "paradox" and solicit explanations from the class:



Suppose a nickel and a dime are firmly attached concentrically, and the nickel is rolled one full turn without slipping along the line $\overleftrightarrow{AB}$. Then $d(A,B)$ is the circumference of the nickel and since $d(A,B) = d(P,Q)$ the circumferences are equal. Aren't they?

Answer. (Don't tell the class too soon.) Of course the circumferences are not equal. If the nickel doesn't slip along $\overleftrightarrow{AB}$ then the dime *must* slip along $\overleftrightarrow{PQ}$.

Challenge Exercises for Sections 5-3, 5-4, 5-5

1. From Figure 5-13, since $d(O,G) = \text{length of } \widehat{FG} = a\phi$, the coordinates of C are $(a\phi, a)$. If $P = (x,y)$ is a point of the locus, then

$$\begin{cases} x = a\phi - b \sin \phi , \\ y = a - b \cos \phi . \end{cases}$$

In Figure 5-14 the point Q less coordinates $(0,k)$. To find k we first find ϕ from $0 = 4\phi - 6 \sin \phi$. We can do this only approximately, from the tables and the fact that $\sin \phi = \frac{2}{3}\phi$. From Table II we have $\sin 1.50 = 0.997$ and $\sin 1.48 = 0.996$. A reasonable estimate gives $\phi \approx 1.50$, within the limits of accuracy of this table. Therefore $k \approx 4 - 6 \cos 1.50$ or $4 - 6(0.071)$. $\therefore$ $Q = (0, 3.57)$.

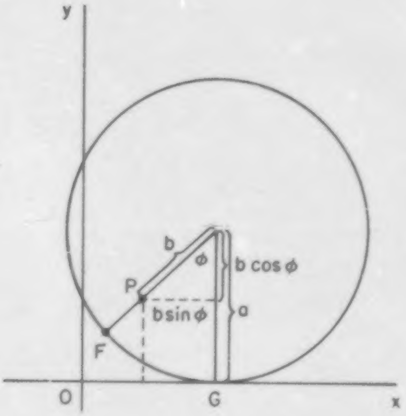
As b gets larger in comparison with a the lower loops get relatively larger, and the graph looks as if it were being compressed horizontally. The lower loops will intesect and overlap and the graph will look more and more like a plane projection of a tight helical spring, or like an elaborate doodle.

2. This drawing should make clear the relations:

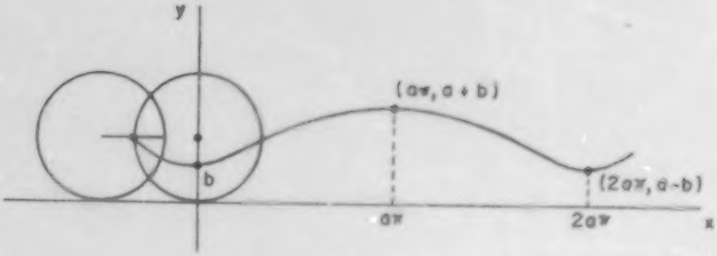
$$\begin{cases} x = d(O,G) - b \sin \phi , \\ y = a - b \cos \phi . \end{cases}$$

The equations for this curtate cycloid are exactly the same as those for the prolate cycloid.

$$\begin{cases} x = a\phi - b \sin \phi , \\ y = a - b \cos \phi . \end{cases}$$



The distinguishing feature for their graphs is in the relative sizes of a and b, as indicated in the text.



3. (Refer to Figure 5-15 in the text.) Since length of $\widehat{AB}$ = length of $\widehat{BP}$, we have $a\phi = b\theta$. Also, $C = \big((a + b)\cos\theta, (a + b)\sin\theta\big)$. If $P = (x,y)$ is a point of the locus then

$$\begin{cases} x = d(O,E) - d(P,D) = (a + b)\cos\theta - a\sin\psi, \\ y = d(C,E) - d(C,D) = (a + b)\sin\theta - a\cos\psi. \end{cases}$$

Since $\theta + \phi + \psi = \frac{\pi}{2}$ we have $\sin\psi = \cos(\theta + \phi)$, and $\cos\psi = \sin(\theta + \phi)$, thus we may eliminate ψ from the equations above and write

$$\begin{cases} x = (a + b)\cos\theta - a\cos(\theta + \phi), \\ y = (a + b)\sin\theta - a\sin(\theta + \phi). \end{cases}$$

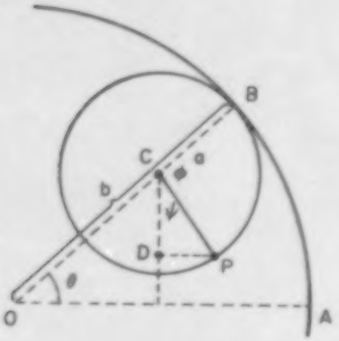
Finally, since $\phi = \frac{b}{a}\theta$ we may eliminate ϕ from the equations above and get

$$\begin{cases} x = (a + b)\cos\theta - a\cos(\theta + \frac{b}{a}\theta), \\ y = (a + b)\sin\theta - a\sin(\theta + \frac{b}{a}\theta). \end{cases}$$

These are usually written

$$\begin{cases} x = (a + b)\cos\theta - a\cos(\frac{a + b}{a}\theta), \\ y = (a + b)\sin\theta - a\sin(\frac{a + b}{a}\theta). \end{cases}$$

4. The analysis here is closely related to that of the previous solution. We furnish a diagram and essential steps only.



$$a\phi = b\theta .$$

$$\phi + \psi - \theta = \frac{\pi}{2}$$

$$d(P,D) = a \sin \psi ,$$

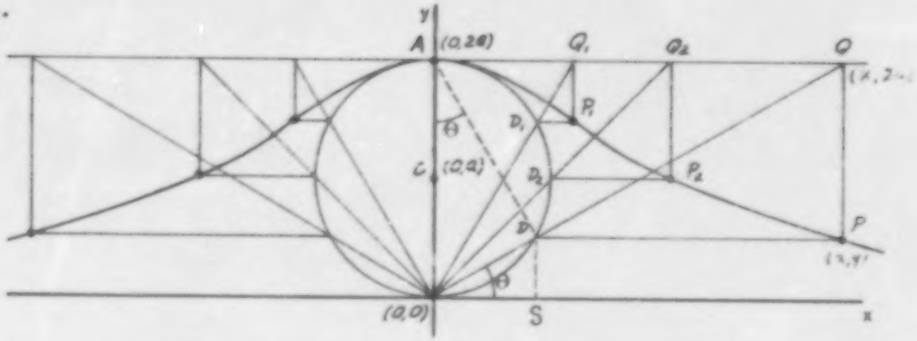
$$d(C,D) = a \cos \psi .$$

$$P = (x,y)$$

$$\begin{cases} x = (b - a) \cos \theta + a \sin \psi , \\ y = (b - a) \sin \theta - a \cos \psi . \end{cases}$$

$$\begin{cases} x = (b - a) \cos \theta + a \cos\left(\frac{b - a}{a}\theta\right), \\ y = (b - a) \sin \theta - a \sin\left(\frac{b - a}{a}\theta\right). \end{cases}$$

5.



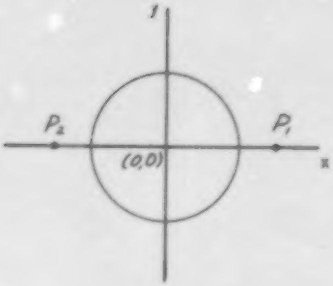
Symmetric in y-axis $0 \le y \le 2a$, x covers all reals asymptotic to x-axis, tangent to $y = 2a$. To get the analytic representation, connect points D, A. Draw $\overline{DS} \perp$ to the x-axis. Then in ($\angle SOD$ ($\angle SOD$) = θ = m($\angle DAO$) ; $y = d(D,S) = d(O,D) \sin \theta$; $d(O,D) = 2a \sin \theta$. Therefore, $y = 2a \sin^2\theta$. Also $x = 2a \cot \theta$. These are parametric equations for the graph.

$$\begin{cases} x = 2a \cot \theta . \\ y = 2a \sin^2\theta . \end{cases}$$

To eliminate the parameter we may square both members of the first equation and then combine with the second to obtain eventually.

$$x^2y = 4a^2(2a - y) , \text{ or } y = \frac{8a^3}{x^2 + 4a^2} .$$

6. Choose coordinate system so that $P_1 = (b,0)$, $P_2 = (-b,0)$. Then we get the condition $x^2 + y^2 = a^2 - b^2$. If $|a| < |b|$, there are no points in locus. If $|a| = |b|$, the locus is the point $(0,0)$. If $|a| > |b|$, the locus is a circle with origin at $(0,0)$ and radius $\sqrt{a^2 - b^2}$.



7. Square $(a,a)(-a,a)(a,-a)(-a,-a)$, constant $4k^2$, $x^2 + y^2 = k^2 - 2a^2$. If $k^2 < 2a^2$, locus is empty set. If $k^2 = 2a^2$, locus is point at $(0,0)$. If $k^2 > 2a^2$, locus is a circle with center $(0,0)$ and radius $\sqrt{k^2 - 2a^2}$.

8. Same square: side $x = a$, $x = -a$, $y = a$, $y = -a$, constant $4k^2$, $x^2 + y^2 = 2k^2 - 2a^2$. If $k^2 < a^2$, locus is empty set. If $k^2 = a^2$, locus is $(0,0)$. If $k^2 > a^2$, locus is circle with center $(0,0)$ and radius $\sqrt{2k^2 - 2a^2}$.

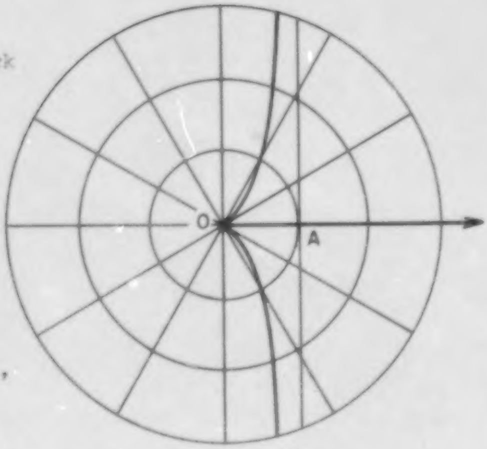
9. $(2c)x + (a + b)y = c(a + b)$ (The sides of the triangle may be extended to allow values of y and x outside of the triangle.)

10. $y^2 + (x - \frac{q}{2})^2 = (\frac{q}{2})^2$. Q does lie on the locus.

11. (Refer to Figure 5-17 in the text.)

$$d(P,S) = d(O,R) = 2a \cos \theta , \qquad \text{from right } \triangle OAR .$$
$$d(O,S) = 2a \sec \theta . \qquad \text{Therefore}$$
$$r = d(O,P) = d(O,S) - d(P,S) = 2a(\sec \theta - \cos \theta) .$$

This is a polar equation for the graph. An equivalent form for this equation is $r = 2a \sin \theta \tan \theta$. To change to rectangular coordinates it is convenient to multiply both members by r and obtain $r^2 = 2a(r \sin \theta)(\tan \theta)$, which yields $x^2 + y^2 = 2a(y)(\frac{y}{x})$, which can be written, $x(x^2 + y^2) = 2ay^2$, or $y^2 = \frac{x^3}{2a - x}$.



The procedure of multiplying both members of the equation by r is convenient, but we must check that the graphs of $r = 2a \sin \theta \tan \theta$ and $r^2 = 2ar \sin \theta \tan \theta$ are the same. The only points that might be on the graph of the latter but not on that of the former are points for which $r = 0$, but the pole, which is the only such point, is already on that graph. The equations therefore do have the same graphs. The idea will escape the students unless they think about such simple examples as $x = y$ and $x^2 = xy$, whose graphs are different. The situation for polar coordinates can be stated as follows. Suppose the pole lies on the graph of the equation $f(r,\theta) = 0$. Then the graphs of that equation and the equation $rf(r,\theta) = 0$ are identical. The same thing can occur when we are dealing with rectangular coordinates. For example, the equations $x^2 = xy$ and $x^3 = x^2y$ have the same graph. The explanation is essentially the same as it was for polar coordinates. All the points which would otherwise have been added to the graph when we multiplied both members of its equation by x, were already points of the graph of $x^2 = xy$.

12. (Refer to Figure 5-18 of the text.)

A polar equation for the locus of R is $r = \frac{a}{\cos \theta}$. Therefore equations for the loci of P and P' are

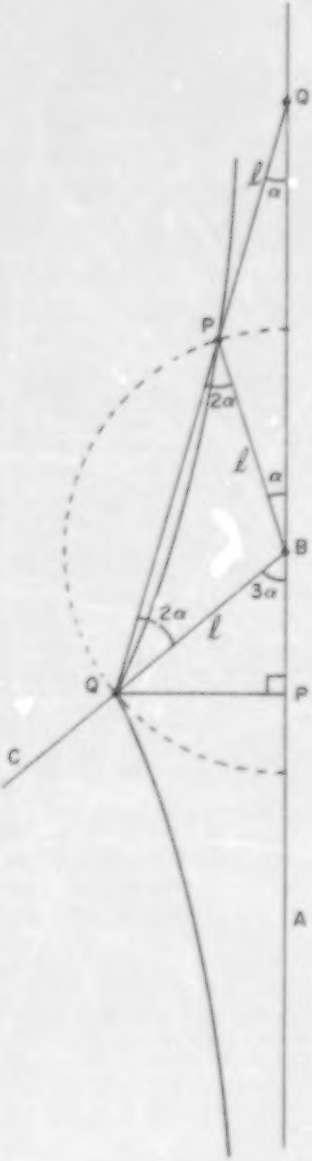
$$r = \frac{a}{\cos \theta} \pm \ell .$$

The trisection of an angle is one of the great classical problems in mathematics under the usual conditions, allowing only compasses and unmarked straightedge, the problem is provably insoluble. (See e.g., What is Mathematics, Courant and Robbins.) However, by the use of special curves which cannot be drawn solely with compasses and unmarked straightedge the problem can be solved. Any such curve used for this purpose is called a trisectrix. To show the use of the conchoid as a trisectrix we proceed as follows:

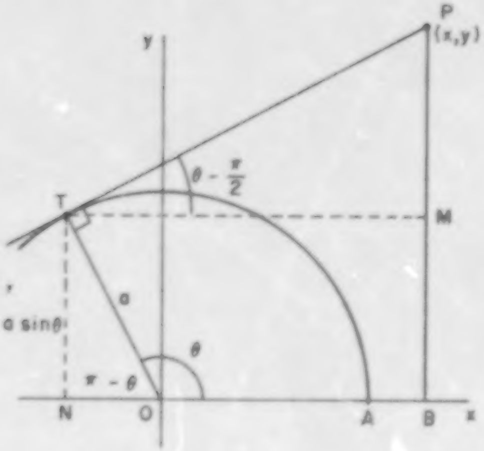
We are given any $\angle ABC$. From O , any point in $\overrightarrow{BC}$, draw $\overline{OR} \perp \overleftrightarrow{AB}$. Construct the left branch of the conchoid as in the text, using $d(O,B)$ as length ℓ . (This is the step which is barred under the classic restriction.) Now construct a circle with B as center, and ℓ as radius, to cut the conchoid at P . Draw $\overrightarrow{OP}$ to cut $\overleftrightarrow{AB}$ at Q . We assert that $m(\angle OQA) = \frac{1}{3} m(\angle OBA)$.

Proof: Draw $\overline{PB}$. Then, from isosceles triangles PQB and PBO we can verify the relations indicated in the diagram.

Note that if ℓ is greater than the distance from the point to the line, then the left branch of the conchoid has a loop, as in the text. If ℓ equals the distance from the point to the line then the left branch has a cusp as in the illustration here. If ℓ is less than the distance from the point to the line, the left branch will have an indentation toward the fixed point.



13. (Refer to Figure 5-19.)



Through T draw lines parallel to the axes as indicated.

$$d(P,T) = a\theta \;;$$

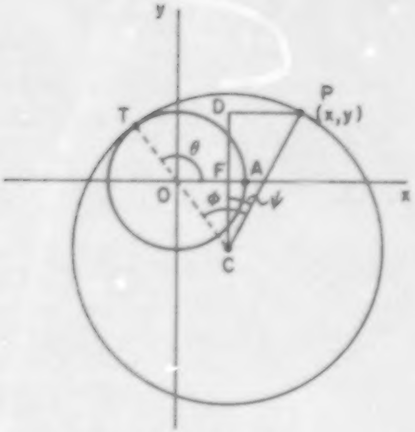
$$\begin{cases} x = d(T,M) - d(O,N) \,, \\ y = d(P,M) + d(T,N) \,. \end{cases}$$

$$\begin{cases} x = a\theta \cos(\theta - \frac{\pi}{2}) - a \cos(\pi - \theta) \,, \\ y = a\theta \sin(\theta - \frac{\pi}{2}) + a \sin \theta \,. \end{cases}$$

Therefore

$$\begin{cases} x = a \cos \theta + a\theta \sin \theta \,, \\ y = a \sin \theta - a\theta \cos \theta \,. \end{cases}$$

14. Students sometimes refer to this problem as the "hula-hoop" problem. Figure 5-20 in the text contains lines which are not pertinent to this solution. Please ignore them and refer to the figure at the right:



$d(C,T) = d(C,P) = b$; $d(O,T) = a$;
$d(O,C) = b - a$.
$\widehat{AT} = \widehat{PT}$, $a\theta = b$, $\phi = \frac{a}{b}\theta$.
$P = (x,y)$.

$$\begin{cases} x = d(O,F) + d(D,P) = (b - a) \sin(\phi - \psi) + b \sin \psi \,, \\ y = d(D,C) - d(F,T) = b \cos \psi - (b - a) \cos(\phi - \psi) \,. \end{cases}$$

Since $\theta = \phi - \psi + \frac{\pi}{2}$, we may eliminate ψ :

$$\sin(\phi - \psi) = \sin(\theta - \frac{\pi}{2}) = -\cos \theta \,,$$

$$\cos(\phi - \psi) = \cos(\theta - \frac{\pi}{2}) = \sin \theta \;;$$

$$\sin \psi = \sin(\phi - \theta + \frac{\pi}{2}) = \cos(\phi - \theta) = \cos(\theta - \phi) \,,$$

$$\cos \psi = \cos(\phi - \theta + \frac{\pi}{2}) = -\sin(\phi - \theta) = \sin(\theta - \phi) \,.$$

Therefore, $\begin{cases} x = -(b - a)\cos\theta + b\cos(\theta - \phi) \, , \\ y = b\sin(\theta - \phi) - (b - a)\sin\theta \, . \end{cases}$

Finally, since $\phi = \frac{a}{b}$,

$$\begin{cases} x = -(b - a)\cos\theta + b\cos\left(\frac{b - a}{b}\theta\right) , \\ y = -(b - a)\sin\theta + b\sin\left(\frac{b - a}{b}\theta\right) . \end{cases}$$

5-6. Parametric Equations of a Straight Line.

The material in this section uses methods developed in this chapter to extend and apply the content introduced in Chapter 2. We recommend here and throughout the book that students be required to refer backwards and forwards. To prepare for this section students should be given, in the preceding few days, some home-work exercises from the latter half of Chapter 2, and that you continue giving some home-work exercises from that chapter as you go on through this section. A systematic overlapping of such assignments is a feature of what is called "spiral" assignments, which we recommend.

The geometric version of the assumption that $x_1 = x_0$ is that the two points are equidistant from the y-axis, the geometric version of the conclusion (that the equations are $x = x_0$, $y = y_0 + mt$) , is that the line through these points is parallel to the y-axis. In the second case the assumption is equivalent to saying that the points are equidistant from the x-axis, and the conclusion is equivalent to saying that the line through them is parallel to the x-axis.

It makes no difference what letter is used for the parameter in parametric equations for a line. Thus we could have represented the lines L_1 and L_2 of Example 2 as follows:

$$L_1 : \begin{array}{l} x = 4 - 2t \\ y = 2 - 6t \end{array} \qquad L_2 : \begin{array}{l} x = -3 - t \\ y = -1 + 3t \end{array}$$

If a student asks whether the two t's are equal, it must be made clear that the question is meaningless. They are both variables and can take any real value. Suppose we had used the representations above and had then tried to find the intersection of the lines by solving the simultaneous equations

$$4 - 2t = -3 - t$$
$$2 - 6t = -1 + 3t.$$

The question we would really have been trying to answer is whether there are any values of t which give the same point on both lines, and this is not the question we started with. This point comes up again in Example 3.

Exercises 5-6

1. (a) $\begin{cases} x = 5 - 3t \\ y = -1 + 4t \end{cases}$ $\quad$ $\begin{cases} x = 2 + 3t \\ y = 3 - 4t \end{cases}$

(b) $\begin{cases} x = 0 + 4t \\ y = 0 + 1t \end{cases}$ $\quad$ $\begin{cases} x = 4 - 4t \\ y = 1 - 1t \end{cases}$

(c) $\begin{cases} x = 2 + 0t \\ y = -3 + 6t \end{cases}$ $\quad$ $\begin{cases} x = 2 - 0t \\ y = 3 - 6t \end{cases}$

(d) $\begin{cases} x = -1 - 5t \\ y = 4 + 0t \end{cases}$ $\quad$ $\begin{cases} x = -6 + 5t \\ y = 4 + 0t \end{cases}$

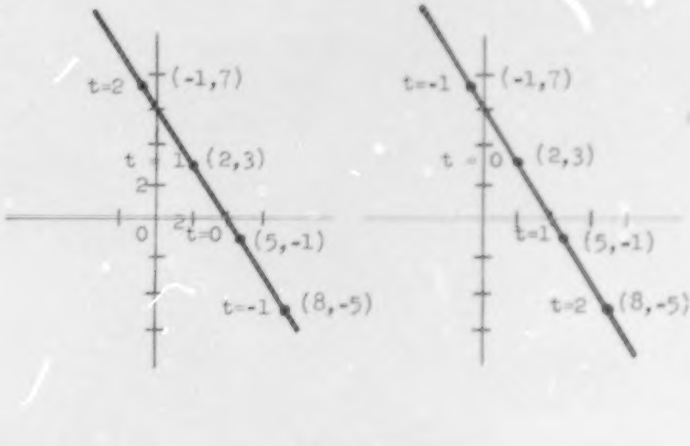
(e) $\begin{cases} x = 1 + 1 \cdot t \\ y = 1 + 1 \cdot t \end{cases}$ $\quad$ $\begin{cases} x = 2 - 1 \cdot t \\ y = 2 - 1 \cdot t \end{cases}$

(f) $\begin{cases} x = -1 + 2t \\ y = -1 + 2t \end{cases}$ $\quad$ $\begin{cases} x = 1 - 2t \\ y = 1 - 2t \end{cases}$

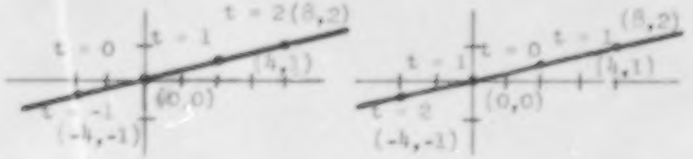
(g) $\begin{cases} x = 1 - 1 \cdot t \\ y = 0 + 1 \cdot t \end{cases}$ $\quad$ $\begin{cases} x = 0 + 1 \cdot t \\ y = 1 - 1 \cdot t \end{cases}$

(h) $\begin{cases} x = 2 - 4t \\ y = -2 + 4t \end{cases}$ $\quad$ $\begin{cases} x = -2 + 4t \\ y = 2 - 4t \end{cases}$

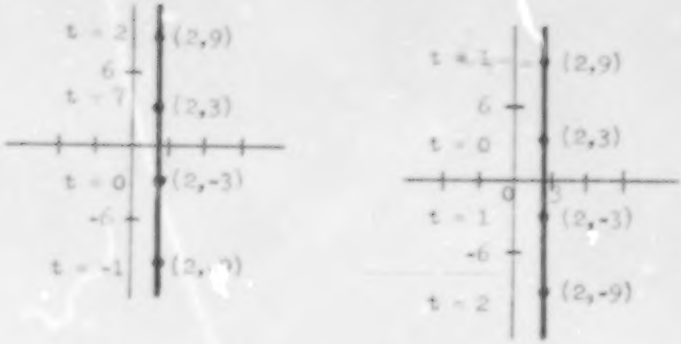
2. (a)



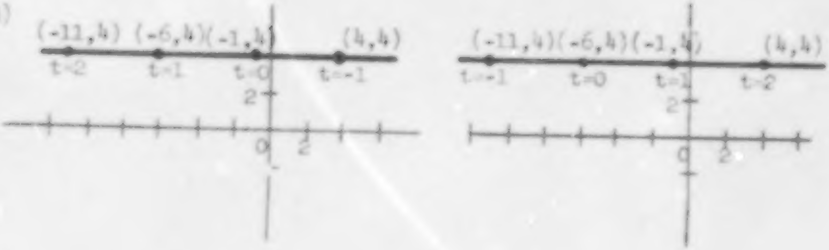
(b)


t = 2 (8,2)
t = 0
t = 1
(4,1)
(0,0)
t = -1
(-4,-1)
(8,2)
t = 1
t = 0
t = 1
(4,1)
(0,0)
t = 2
(-4,-1)

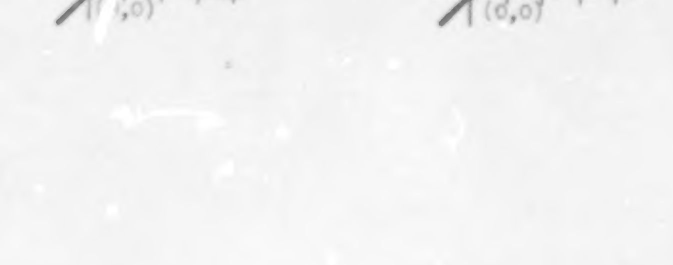
(c)


t = 2
(2,9)
6
t = 7
(2,3)
t = 0
(2,-3)
-6
t = -1
t = 1
(2,9)
6
t = 0
(2,3)
0
3
t = 1
(2,-3)
-6
t = 2
(2,-9)

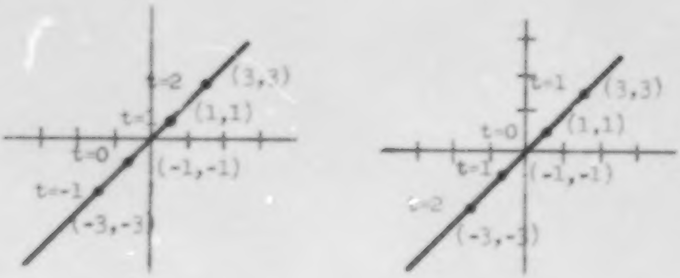
(d)


(-11,4)
(-6,4)
(-1,4)
(4,4)
t=2
t=1
t=0
t=-1
2
0
2
(-11,4)
(-6,4)
(-1,4)
(4,4)
t=-1
t=0
t=1
t=2
2
0
2

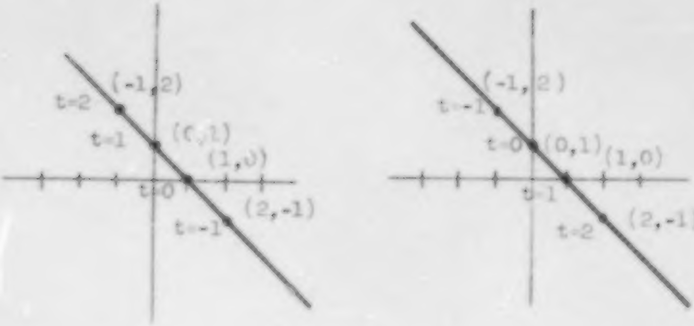
(e)


t=2
(3,3)
t=1
(2,2)
t=0
(1,1)
t=-1
(0,0)
t=-1
(3,3)
t=0
(2,2)
t=1
(1,1)
t=2
(0,0)

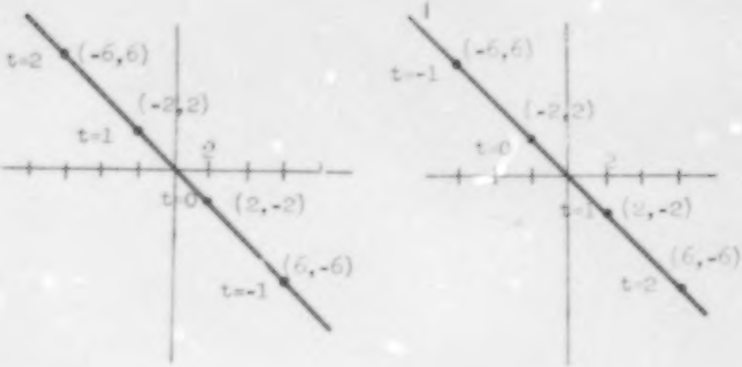
(f)



(g)



(h)



3. (a) $(-14,21)$
   (b) The lines are parallel; their pairs of direction numbers are equivalent: $(6,-4) = (-2(-3), -2(2))$
   (c) The lines are coincident; their pairs of direction numbers are equivalent and they have at least one point $(-3,2)$ in common.

4. Using points $(1,1)(4,3)$ on the line $L : 2x - 3y + 1 = 0$.

$$x = 1 + 3t$$
$$y = 1 + 2t .$$

5. $x_1 - x_2 = \ell(t_1 - t_2), \ y_1 - y_2 = m(t_1 - t_2)$

$$d(P_1, P_2) = \sqrt{\ell^2(t_1 - t_2)^2 + m^2(t_1 - t_2)^2}$$
$$= \sqrt{(t_2 - t_1)^2} \cdot \sqrt{\ell^2 + m^2}$$
$$= |t_2 - t_1| \cdot \sqrt{\ell^2 + m^2}$$

6. $\begin{cases} x = 16 + t(-24) \\ y = 2 + t(10) \end{cases}$

7. (a) Substituting $x = \lambda t$, $y = \mu t$ into $ax^2 + by^2 = a^2b^2$ gives

$$a\lambda^2 t + b\mu^2 t^2 = a^2b^2$$
$$t^2(a\lambda^2 + b\mu^2) = a^2b^2$$

If $$a\lambda^2 + b\mu^2 \neq 0$$
$$t^2 = \frac{a^2b^2}{a\lambda^2 + b\mu^2} ;$$

if $$a\lambda^2 + b\mu^2 > 0$$
$$t = \pm \frac{|ab|}{\sqrt{a\lambda^2 + b\mu^2}} ;$$

hence line intersects figure at points equidistant from 0 under conditions mentioned.

(b) Putting $x = \lambda t$, $y = \mu t$ into $y = ax^3$, we get $t = a\lambda^3 t^3$. If $a > 0$ for $\mu \neq 0$, $\lambda \neq 0$ and considering only $t \neq 0$ we get $t^2 = \frac{\mu}{a\lambda^3}$.

If $\mu \cdot \lambda > 0$, $t = \pm \frac{1}{\lambda} \sqrt{\frac{\mu}{a\lambda}}$ and intersections are symmetric.

If $\mu \cdot \lambda < 0$, there are no intersections for $t \neq 0$.

Thus the origin is the center.

(2) $a < 0$, for $\mu \neq 0$, $\lambda \neq 0$ and considering $t \neq 0$ we get

$$t^2 = \frac{\mu}{a\lambda^3} .$$

If $\mu \cdot \lambda > 0$, there are no intersections for $t \neq 0$.

If $\mu \cdot \lambda < 0$, then there are intersections for

$$t = \pm \frac{1}{\lambda} \sqrt{\frac{\mu}{a\lambda}} .$$

Again the origin is the center.

(c) Putting $x = \lambda t$, $y = \mu t$ into $y = \dfrac{x^3}{x^2 - 1}$

we get $\mu t = \dfrac{\mu^3 t^3}{\lambda^2 t^2 - 1}$ which is not defined for $\lambda t = 1$

$$\mu\lambda^2 t^3 - \mu t = \mu^3 t^3 .$$

If $\mu \neq 0$ $\qquad \lambda^2 t^3 - t = \mu^2 t^3$;

if $t \neq 0$ $\qquad \lambda^2 t^2 - 1 = \mu^2 t^2$;

$$\therefore \quad t^2(\lambda^2 - \mu^2) = 1 .$$

If $\mu^2 \neq \lambda^2$ $\qquad t^2 = \dfrac{1}{\lambda^2 - \mu^2}$.

If $\lambda^2 > \mu^2$, then the line intersects the curve for

$$t = \pm\sqrt{\frac{1}{\lambda^2 - \mu^2}} \text{ , that is, symmetrically.}$$

There is no value of t if $\lambda^2 \leq \mu^2$. Thus the curve has the origin as its center.

8. We suppose that a bounded set S has two centers, and show that we get a contradiction. We call these centers O and I and establish a coordinate system with origin at O, with x-axis along $\overleftrightarrow{OI}$, and I as the point $(1,0)$. If O and I are centers then O has a symmetric image, O_1 in I, and $O_1 = (2,0)$. O_1 has a symmetric image O_2 in O_1 and $O_2 = (-2,0)$. O_2 has a symmetric image O_3 in I, and $O_3 = (3,0)$, and so on. The points $O_1, O_3, O_5, \ldots$, are all members of S and their coordinates, $(2,0), (3,0), (4,0), \ldots$, indicate that they are farther and farther from the origin. Clearly they cannot all be enclosed by an finite rectangle, which means that S cannot be bounded.

The statement is not true for unbounded sets; for example any point of a line is a center of the set of points of that line.

9. We express the line in parametric form using direction cosines:

$$\begin{cases} x = 5 + 0.8t , \\ y = 8 + 0.6t . \end{cases}$$

When $t = 1$, $(x,y) = (5.8 , 8.6)$;
when $t = -1$, $(x,y) = (4.2 , 7.4)$.

10. $$\begin{cases} x = 0 + 3t , \\ y = 9 + 4t . \end{cases}$$

It is simplest here to use $d(A,B)$ units along the line. When $t = 5$, $(x,y) = (15,29)$; when $t = -5$, $(x,y) = (-15,-11)$.

Review Exercises

In the answers to these exercises we supply, in most cases, the simplest and most directly achieved answer. It is always to be understood that a given graph has infinitely many analytic representations. Some of these may be trivially related as: $y = 5$ and $2y = 10$; some non-trivally as: $x + 2y - 11 = 0$ and

$$\begin{cases} x = 5 + 4t , \\ y = 3 - 2t . \end{cases}$$

The teacher is particularly urged in this chapter to consider carefully any pupil's answer which may differ from the one presented here. It may be correct, but written in unfamiliar form, and the student may, with benefit, carry the burden of showing the equivalence of the two.

When we are asked for an analytic description of a set, for example, 2(a) below, we will usually write our answer in the form in which it appears in the literature:

$$x - 4y + 7 = 0 ,$$

instead of the longer form:

$$\{(x,y) : x - 4y + 7 = 0\} .$$

1. (a) The lines: $y = x$ and $y = -x$; or $y^2 = x^2$.
   (b) The line: $x = 8$.
   (c) The line: $y = 4$.
   (d) The line: $3x - 4y - 8 = 0$.
   (e) The circle: $(x - 5)^2 + (y - 8)^2 = 9$, which can also be written:
   $x^2 + y^2 - 10x - 16y + 80 = 0$.
   (f) The lines: $x = 2$ and $x = 8$.

(g) The lines: $y = 1$ and $y = -5$.

(h) The lines: $3x - 4y + 22 = 0$ and $3x - 4y - 8 = 0$.

(i) The lines: $x = k + h$ and $x = k - h$.

(j) The lines: $y = q + p$, $y = q - p$.

(k) If $ax + by + c = 0$ represents a line, then $a^2 + b^2 \neq 0$ and the distance from $P = (x_0,y_0)$ to this lines is given by

$d = \dfrac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$. This equation is equivalent to

$ax_0 + by_0 + c = \pm d\sqrt{a^2 + b^2}$, therefore the locus of all such points $P = (x,y)$ is the pair of lines represented by

$ax + by + c + d\sqrt{a^2 + b^2} = 0$, and $ax + by + c - d\sqrt{a^2 + b^2} = 0$.

(l) The distance from $P = (x,y)$ to $A = (5,0)$ is $\sqrt{(x - 5)^2 + y^2}$; and to $B = (11,0)$ is $\sqrt{(x - 11)^2 + y^2}$. The condition is equivalent to; $\sqrt{(x - 5)^2 + y^2} = 2\sqrt{(x - 11)^2 + y^2}$. This equation is an answer to the exercise, but it can be written more simply as $x^2 + y^2 - 26x + 143 = 0$, or as $(x - 13)^2 + y^2 = 4^2$. This last equation yields the additional information that the graph is a circle with center at $(13,0)$ and with radius 4 .

(m) The condition yields directly: $y = \sqrt{(x - 5)^2 + (y - 8)^2}$ or more simply $x^2 - 10x - 16y + 89 = 0$. This can also be written $(x - 5)^2 = 16(y - 4)$, which can be interpreted to be an equation of a parabola with vertex at $(5,4)$, axis along the y-axis, and open upward.

(n) As above, we get the parabola: $y^2 - 8x + 24 = 0$.

(o) The distance from $P = (x,y)$ to $D = (5,3)$ is $\sqrt{(x - 5)^2 + (y - 3)^2}$. The distance from $P = (x,y)$ to the line $3x - 4y + 7 = 0$ is $\dfrac{|3x - 4y + 7|}{\sqrt{3^2 + 4^2}}$. An answer to this exercise is given by the statement of equality for these two distances,

$\sqrt{(x - 5)^2 + (y - 3)^2} = \dfrac{|3x - 4y + 7|}{\sqrt{3^2 + 4^2}}$. This can be written somewhat more simply as $16x^2 + 24 + y + 9y^2 - 292x - 94y + 801 = 0$. We state that the graph is a parabola with an oblique axis perpendicular to the given line, but we leave any further discussion of this equation and graph for Chapter 10.

(p) As in the previous exercise, an answer is given by:

$$\sqrt{(x - r)^2 + (y - s)^2} = \frac{|ax + by + c|}{\sqrt{a^2 + b^2}}$$, which can be written also as:

$$(ax + by + c)^2 = (a^2 + b^2)\ (x - r)^2 + (y - s)^2$$, or, as a polynomial in x and y :

$$b^2x^2 - 2abxy + a^2y^2 - 2(ac + a^2r + b^2r)x - 2(bc + a^2s + b^2s)y$$
$$+ (a^2r^2 + a^2s^2 + b^2r^2 + b^2s^2 - c^2) = 0 \ .$$

We state again without proof that the graph of this equation is a parabola with its axis perpendicular to the given line.

2. In (a) - (i) we give our answers in both rectangular and parametric forms; either or both may be used.

(a) $x - 4y + 7 = 0$; or $\begin{cases} x = -3 + 8t \ , \\ y = 1 + 2t \ . \end{cases}$

(b) $x - 4y + 7 = 0 \ , \ x \geq -3$; or $\begin{cases} x = -3 + 8t \ , \\ y = 1 + 2t \ , \end{cases}$ $t \geq 0$.

(c) $x - 4y + 7 = 0 \ , \ -3 \leq x \leq 5$; or $\begin{cases} x = -3 + 8t \ , \\ y = 1 + 2t \ , \end{cases}$ $0 \leq t \leq 1$.

(d) $x + 2y - 11 = 0$; or $\begin{cases} x = 5 - 4t \ , \\ y = 3 + 2t \ . \end{cases}$

(e) $x + 2y - 11 = 0 \ , \ x \leq 5$; or $\begin{cases} x = 5 - 4t \ , \\ y = 3 + 2t \ , \end{cases}$ $t \geq 0$.

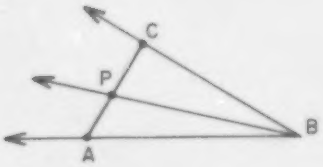
(f) $x + 2y - 11 = 0 \ , \ 1 \leq x \leq 5$; or $\begin{cases} x = 5 - 4t \ , \\ y = 3 + 2t \ , \end{cases}$ $0 \leq t \leq 1$.

(g) $x - y + 4 = 0$; or $\begin{cases} x = 1 - 4t \ , \\ y = 5 - 4t \ . \end{cases}$

(h) $x - y + 4 = 0 \ , \ x \leq 1$; or $\begin{cases} x = 1 - 4t \ , \\ y = 5 - 4t \ , \end{cases}$ $t \geq 0$.

(i) $x - y + 4 = 0 \ , \ -3 \leq x \leq 1$; or $\begin{cases} x = 1 - 4t \ , \\ y = 5 - 4t \ , \end{cases}$ $0 \leq t \leq 1$.

(j) This, and the next four parts of this exercise are most readily done with parametric representations or vectors. The interior of $\angle ABC$ can be described as the set of points of the interior of all rays $\overrightarrow{BP}$, where P is a point of the interior of $\overline{CA}$. In that case $P = (x,y)$; where $x = 1 - 4t$, $y = 5 - 4t$, $0 < t < 1$, from (i) above. We need another parameter to give us the interior of $\overrightarrow{BP}$. Thus direction numbers for $\overrightarrow{BP}$ are $(1 - 4t - 5 , 5 - 4t - 3)$, or $(-4 - 4t , 2 - 4t)$. Thus, for a point $Q = (x,y)$ of the interior of $\overrightarrow{BP}$ we have $x = 5 + s(-4 - 4t)$, $y = 3 + s(2 - 4t)$, $s > 0$. We present this answer more neatly:
$\{(x,y) : x = 5 - 4x - 4st , y = 3 + 2s - 4st , s > 0 , 0 < t < 1\}$.



In vector form, if P is an interior point of $\overline{CA}$ then $\overrightarrow{p} = \overrightarrow{c} + t(\overrightarrow{a} - \overrightarrow{c})$, $0 < t < 1$. If Q is an interior point of $\overrightarrow{BP}$, then $\overrightarrow{q} = \overrightarrow{b} + s(\overrightarrow{p} - \overrightarrow{b})$, $s > 0$. In terms of $\overrightarrow{a}$, $\overrightarrow{b}$, $\overrightarrow{c}$, we have $\overrightarrow{q} = \overrightarrow{b} + s\left(\overrightarrow{c} + t(\overrightarrow{a} - \overrightarrow{c}) - \overrightarrow{b}\right)$, $\overrightarrow{q} = (st)\overrightarrow{a} + (1 - s)\overrightarrow{b} + (s - st)\overrightarrow{c}$, with $s > 0$, $0 < t < 1$. Note that the sum of the scalar multipliers is 1 .

We can show the equivalence of the vector and parametric forms by expressing each vector in terms of its components and then combining, retaining the parametric conditions $s > 0$, $0 < t < 1$. Thus: $\overrightarrow{q} = [x,y]$, $\overrightarrow{a} = [-3,1]$, $\overrightarrow{b} = [5,3]$, $\overrightarrow{c} = [1,5]$. Then
$[x,y] = st[-3,1] + (1 - s)[5,3] + (s - st)[1,5]$,
$[x,y] = [-3st + 5 - 5s + s - st , st + 3 - 3s + 5s - 5st]$,
$[x,y] = [5 - 4s - 4st , 3 + 2s - 4st]$.
Therefore

$$\begin{cases} x = 5 - 4s - 4st , \\ y = 3 + 2s - 4st ; \end{cases}$$

and these are the parametric equations we found before.

(k) If P is a point of the interior of $\overline{AB}$, then $P = (-3 + 8t , 1 + 2t)$, $0 < t < 1$. Proceed as in the previous solution and obtain the answer,
$\{(x,y) : x = 1 - 4s + 8st , y = 5 - 4s + 2st , s > 0 , 0 < t < 1\}$.
In vector form $\overrightarrow{p} = \overrightarrow{a} + t(\overrightarrow{b} - \overrightarrow{a})$, $0 < t < 1$, and $\overrightarrow{q}$, the vector to any point Q of the interior of $\angle BCA$ is given by

$\overrightarrow{q} = \overrightarrow{c} + s(\overrightarrow{p} - \overrightarrow{c})$, $s > 0$. This can be written in terms of $\overrightarrow{a}$, $\overrightarrow{b}$, $\overrightarrow{c}$ as was done in the previous solution:

$\overrightarrow{q} = (s - st)\overrightarrow{a} + (st)\overrightarrow{b} + (1 - s)\overrightarrow{c}$, $s > 0$, $0 < t < 1$.

Note the resemblance to the result in the previous exercise. The component forms of these vectors can be used to relate this result to the parametric equation found a few lines earlier.

(ℓ) (Refer to the two previous solutions.)

$\overrightarrow{p} = \overrightarrow{c} + t(\overrightarrow{b} - \overrightarrow{c})$, $0 < t < 1$; $\overrightarrow{q} = \overrightarrow{a} + s(\overrightarrow{p} - \overrightarrow{a})$, $s > 0$.

$\overrightarrow{q} = (1 - s)\overrightarrow{a} + (st)\overrightarrow{b} + (s - st)\overrightarrow{c}$, $s > 0$, $0 < t < 1$.

The parametric form is

$\{(x,y) : x = -3 + 4s + 4st , 1 + 4s - 2st , s > 0 , 0 < t < 1\}$.

(m) The interior of $\triangle ABC$ is part of the interior of $\angle ABC$. If we refer to the solution of part (j) of this group we need now use only the interior points of $\overline{BP}$ where P is an interior point of $\overline{AC}$. We can effect this result by a simple change on the parameter s which we now take $0 < s < 1$. Our solution in vector form is therefore:

$\overrightarrow{q} = (st)\overrightarrow{a} + (1 - s)\overrightarrow{b} + (s - st)\overrightarrow{c}$, with $0 < s < 1$, $0 < t < 1$.

We could use the results of (k) and (ℓ) above, and obtain

$\overrightarrow{q} = (s - st)\overrightarrow{a} + (st)\overrightarrow{b} + (1 - s)\overrightarrow{c}$, $0 < s < 1$, $0 < t < 1$;

$\overrightarrow{q} = (1 - s)\overrightarrow{a} + (st)\overrightarrow{b} + (s - st)\overrightarrow{c}$, $0 < s < 1$, $0 < t < 1$.

The similarity of these expressions leads to a more symmetric formula, if we note that the scalar multipliers are non-negative and have the sum 1 . We may write a vector formula for the interior of $\triangle ABC$ thus:

$\overrightarrow{q} = \alpha\overrightarrow{a} + \beta\overrightarrow{b} + \gamma\overrightarrow{c}$, where α , β , γ are non-negative and $\alpha + \beta + \gamma = 1$.

(n) $x + 2y + 1 = 0$.

(o) $x - y - 2 = 0$.

(p) $x - 4y + 19 = 0$.

(q) $2x - y + 7 = 0$.

(r) $x + y - 8 = 0$.

(s) $4x + y - 9 = 0$.

(t) $x - 2y + 5 = 0$.

(u) $y = 3$.

(v) $x = 1$.

(w) The line $y = 1$ is parallel to the x-axis, and the line $x = -3$ is parallel to the y-axis.

(x) $4x + y - 6 = 0$.

(y) $2x - y - 2 = 0$.

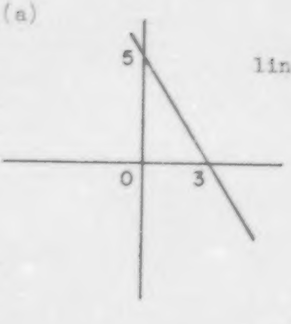
(z) If the center of the circle is at (u,v) then

$$(1-u)^2+(5-v)^2 = (5-u)^2 + (3-v)^2 = (3+u)^2 + (1-v)^2 = r^2$$

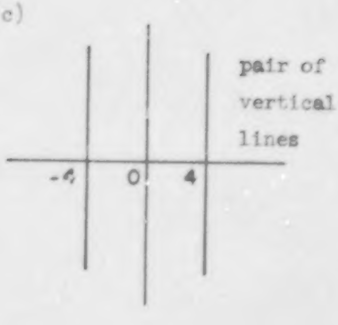
Solving these equations gives the coordinates of the center, $(\frac{4}{3}, \frac{2}{3})$, and the length of the radius, $\frac{\sqrt{170}}{3}$. Thus the circle has the equation, $(x - \frac{4}{3})^2 + (y - \frac{2}{3})^2 = \frac{170}{9}$, which may be written also as $3x^2 + 3y^2 - 8x - 4y - 50 = 0$.

3. The abbreviated sketch we supply for each part of this exercise should indicate the answers requested originally. Other brief comments are supplied as seem necessary.

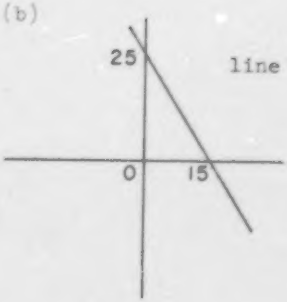
(a)



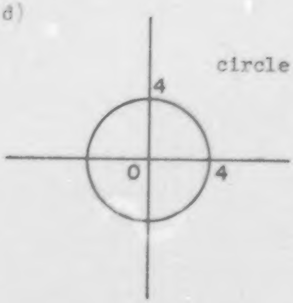
(c)



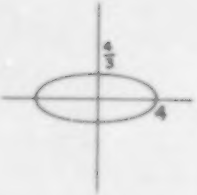
(b)



(d)

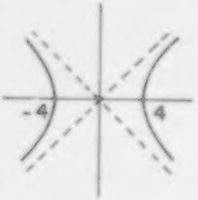


(e)



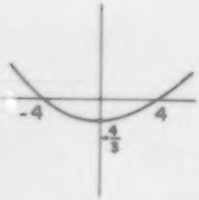
ellipse

(f)



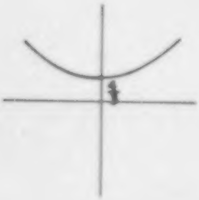
hyperbola

(g)



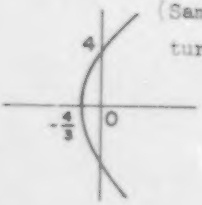
parabola

(h)



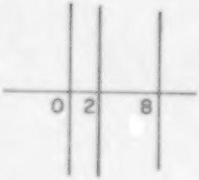
parabola

(i)



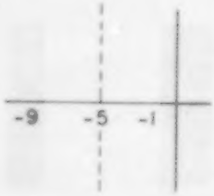
(Same as (g) turned 90°)

(j)



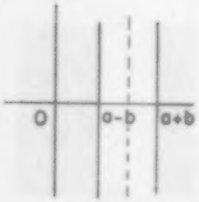
Pair of
vertical
lines

(k)



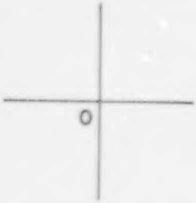
The region
between but
not including
the vertical
lines.

(ℓ)



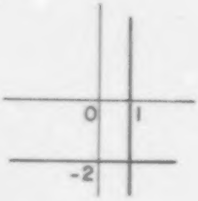
The entire
plane except.

(m)



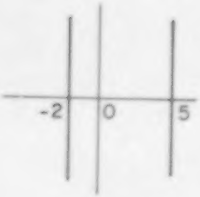
The x- and
y-axes.

(n)



The two lines
indicated
$x = 1$, and
$y = -2$.

(o)

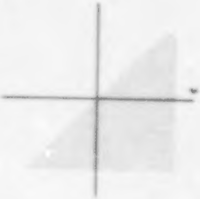


$x^2 - 3x - 10 = 0$

$(x - 5)(x + 2) = 0$

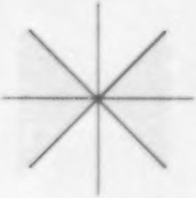
The pair of vertical lines.

(p)

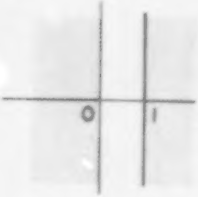


The region below the line $y = x$.

(q)



The shaded region between the lines $y = \pm x$, as shown .



$x^2 < x^2$ is equivalent to $x^2 - x > 0$, or $x(x - 1) > 0$.

This inequality is true for all x except for $0 \leq x \leq 1$. The graph is the entire plane except region between the vertical lines.

4. We do not supply full answers here, but only enough in sketch or comment to make contact with familiar material.

   (a) Circle with radius 3 and center at the pole.

   (b) The interior of the circle in (a) above.

   (c) Since there is no negative restriction on r , the set is the entire plane. If $0 < r < 3$ the set would be the same as (b) above.

   (d) The plane outside the circle of (a) above.

   (e) The line through the pole making with the polar axis an angle of measure 2 .

   (f) Since there is no negative restriction on θ the set is the entire plane.

   (g) If $r > 0$ the graph is a spiral similar to that of Figure 5-5 but opening more rapidly. It contains the pole and crosses the polar axis to the right at 4π , 8π , 12π , ... , and to the left at (abscissas) -2π , -6π , -10π , ... If $r < 0$ the graph is the symmetric image with respect to the pole of the path just described, thus the entire graph is a double spiral opening counterclockwise and crossing the polar axis at (abscissas) 0 , 2π , -2π , 4π , -4π, 6π , -6π , ...

(h) The entire plane. Compare the polar and rectangular conditions: $x = y$ gives a line, and $x < y$ a half-plane; $r = \theta$ a spiral, and $r < \theta$ the whole plane.

(i) Two lines through the origin, $\theta = 2.1$ and $\theta = 1.9$.

(j) The annular region between two concentric circles of radii 4.9 and 5.1 with centers at the pole.

In the next few solutions we supply a familiar equivalent equation in rectangular coordinates related in the obvious way, to polar coordinates. The graphs for parts (k) ... (q) are all lines, and in each case the absolute value of the numerator is the distance from the pole to the line.

(k) The line $y = 6$.

(ℓ) The line $x = -3$.

(m) The line $x = -2$.

(n) The line $x = 5$.

(o) The line through $(\sqrt{2},0)$ with slope 1.

(p) The line through $(-4\sqrt{2},0)$ with slope 1.

(q) We take $0 \leq b \leq 2\pi$. If $b = 0$ the graph is the line $y = a$; if $b = 2\pi$ the graph is the line $y = -a$. If $b = \frac{\pi}{2}$ or $\frac{3\pi}{2}$ the graph is the line $x = -a$ or $x = a$, respectively. If b has any other value in the indicated domain the graph is the line through $(-a \csc b, 0)$, with the slope $\tan b$.

(r) Polar inequalities must be carefully analyzed. In this case if $0 < \theta < \pi$ the graph is the region above the line $y = 1$. If $\theta = \pi$ there is no value of r for which $r > \frac{1}{\sin \theta}$ since $\frac{1}{\sin \theta}$ is not defined then. If $\pi < \theta < 2\pi$ then the graph contains every point which is below the line $y = 1$ and on any line which intersects the line $y = 1$ and which goes through the origin. That is, this part of the graph is the region below the line $y = 1$, excluding the two half-lines along the x-axis: $y = 0$, $x > 0$, and $y = 0$, $x < 0$. To summarize, the graph of $r > \frac{1}{\sin \theta}$ is the entire plane except the points of the line $y = 1$ and the points of the two half-lines along the x-axis: $y = 0$, $x > 0$, and $y = 0$, $x < 0$. It is instructive to investigate, but we will not, the relation between $r < \frac{1}{\sin \theta}$ and $r \sin \theta < 1$, noting that this second inequality is related to $y < 1$.

(s) We consider $0 \leq \theta < 2\pi$. If $\theta = 0$ the graph is that part of the x-axis to the left of $x = 2$. If $0 < \theta < \frac{\pi}{2}$ we get, for $0 < r < \frac{2}{\cos \theta}$, the vertical strip above the x-axis and between the y-axis and the line $x = 2$. For this same domain, if $r \leq 0$ we get the origin and all points in the third quadrant. If $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$ we get the region to the right of the line $x = 2$. Since $\frac{2}{\cos \theta}$ is not defined for $\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$ there is no value of r defined for these values of θ . If $\frac{3\pi}{2} < \theta < 2\pi$ and $0 < r < \frac{2}{\cos \theta}$ we get the vertical strip below the x-axis and between the y-axis and the line $x = 2$. For this same domain if $r \leq 0$ we get the origin and all points in the second quadrant. To summarize, the graph we want is the entire plane except the line $x = 2$, and the two half-lines along the y-axis; $x = 0$, $y > 0$, and $x = 0$, $y < 0$. It is instructive to investigate, though we will not, the relation between $r > \frac{2}{\cos \theta}$ and $r \cos \theta > 2$, noting that this second inequality is related to $x > 2$.

(t) The pole.

5. In the discussion of related polar equations in Section 5-2 we used the fact that the point $P = (r,\theta)$ has also the coordinates $(-r , \theta + \pi)$. Thus, if P is on the graph of $r = f(\theta)$ we must also have P on the graph of $-r = f(\theta + \pi)$. Then we obtained the equivalent equation $r = -f(\theta + \pi)$, but this step cannot be carried through so easily with inequalities. If the point (r,θ) is on the graph of $r > f(\theta)$, then that same point, now indicated by $(-r,\theta + \pi)$, is on the graph of $-r > f(\theta + \pi)$, but this last inequality is equivalent to $r < -f(\theta + \pi)$, and this is the related polar inequality of $r > f(\theta)$. However, the original inequality can frequently be written in the form $g(r,\theta) > 0$ for which the related polar inequality is $g(-r, \theta + \pi) > 0$ and is usually easier to handle.

(a) $r^2 = 9$

(b) $r^2 < 9$

(c) $r > -3$

(d) $r < -3$

(e) $\theta = 2 - \pi$

(f) $\theta < -\frac{\pi}{2}$

(g) $r = -2(\theta + \pi)$

(h) $r > -(\theta + \pi)$

(i) $|\theta + \pi - 2| = .1$

(j) $|-r - 5| < .1$, or $|r + 5| < .1$

(k) $r = \frac{6}{\sin \theta}$

(ℓ) $r = \frac{-3}{\cos \theta}$

(m) $r = \frac{-2}{\cos \theta}$

(n) $r = \frac{5}{\cos \theta}$

(o) $r = \frac{1}{\cos(\theta + \frac{\pi}{4})}$

(q) $r = \frac{a}{\sin(\theta - b)}$

(r) $r < \frac{1}{\sin \theta}$

(s) $r > \frac{2}{\cos \theta}$

(t) $r = 0$

6. (a) $y = x^2 - 2x + 2$

(b) $x - 2y + 4 = 0$

(c) $2y = x + xy$

(d) $x^3 = y^2 + xy$

(e) $y = x^2 - 2$

(f) $\frac{x^2}{9} + \frac{y^2}{16} = 1$

(g) $\frac{(x - 2)^2}{9} + \frac{(y - 4)^2}{25} = 1$

(h) $4y^2 = x^2(4 - x^2)$

(i) $\frac{1}{x^2} + \frac{1}{y^2} = 1$

(j) $x^2 = 16y^2(1 - y^2)(1 - 2y^2)^2$

7. $x = 3 - \frac{3}{5}t$,

$y = 7 - \frac{4}{5}t$.

8. $x = 84t$,
$y = 288t$.

9. When $t = 3$, $A = (8,0)$, $B = (-1,14)$, $d(A,B) = \sqrt{277}$.
When $t = 5$, $A = (14,-2)$, $B = (-5,16)$, $d(A,B) = \sqrt{685}$.

10. When $t = 2$ $P_1 = (x_1 + 2\ell_1, y_1 + 2m_1)$, $P_2 = (x_2 + 2\ell_2, y_2 + 2m_2)$,

$$d(P_1P_2) = \sqrt{(x_1 - x_2 + 2\ell_1 - 2\ell_2)^2 + (y_1 - y_2 + 2m_1 - 2m_2)^2} .$$

11. (a) $x = \cos(\frac{\pi}{2} + 6\pi t)$, $y = \sin(\frac{\pi}{2} + 6\pi t)$.

(b) $x = \cos(-\frac{\pi}{2} - 4\pi t)$, $y = \sin(-\frac{\pi}{2} - 4\pi t)$.

(c) $x = \cos(-\frac{\pi}{6} + 2\pi t)$, $y = \sin(-\frac{\pi}{6} + 2\pi t)$.

(d) $x = \cos(\pi - 8\pi t)$, $y = \sin(\pi - 8\pi t)$.

(e) $x = \cos(\frac{7\pi}{6} + \pi t)$, $y = \sin(\frac{7\pi}{6} + \pi t)$.

12. We give the time in seconds and the angular position in terms of θ only. The rectangular coordinates of the position are $(\cos\theta, \sin\theta)$.

(a) $\frac{1}{10}$, $(\frac{11\pi}{10})$
(b) $\frac{2}{3}$, $(\frac{\pi}{2})$
(c) $\frac{1}{28}$, $(\frac{5\pi}{7})$
(d) $\frac{2}{15}$, $(\frac{13\pi}{10})$
(e) $\frac{5}{18}$, $(\frac{7\pi}{18})$
(f) $\frac{3}{8}$, (0)
(g) $\frac{1}{15}$, $(\frac{37\pi}{30})$
(h) $\frac{7}{60}$, $(\frac{\pi}{15})$
(i) $\frac{4}{3}$, $(\frac{\pi}{2})$
(j) $\frac{11}{54}$, $(\frac{37\pi}{27})$

13. Assume that it starts from its farthest right position

$$\begin{cases} x = 4 + 3\cos 4\pi t , \\ y = 5 + 3\sin 4\pi t . \end{cases}$$

If, when $t = 0$ it starts from the angular position θ relative to its center, then the equations of motion are

$$\begin{cases} x = 4 + 3\cos(4\pi t + \theta) , \\ y = 5 + 3\sin(4\pi t + \theta) . \end{cases}$$

14. Assume it starts from the angular position θ relative to its center. Then

$$\begin{cases} x = -1 + 2\cos(\theta - 2\pi t) , \\ y = \sin(\theta - 2\pi t) . \end{cases}$$

15. These are all circular paths with center at the center of the clock. We give the radius, angular position of starting point, direction of rotation, and angular velocity in revolutions per minute.

(a) 4 , 0 , counterclockwise, 2 rpm.

(b) 6 , $\frac{\pi}{2}$, counterclockwise, 3 rpm.

(c) 10 , [illegible] , [illegible] 5 rpm.

(d) 8 , π , [illegible]ockwise, 2 rpm.

(e) The given equations are equivalent to

$$\begin{cases} x = 2 \cos (\frac{\pi}{2} - 2\pi t) , \\ y = 2 \sin (\frac{\pi}{2} - 2\pi t) ; \end{cases}$$

therefore the motion is as above: 2 , $\frac{\pi}{2}$, clockwise, 1 rpm.

16. (a) $\begin{cases} x = 5 \cos \theta , \\ y = 3 \sin \theta . \end{cases}$

(b) $\begin{cases} x = 3 \cos \theta , \\ y = 4 \sin \theta . \end{cases}$

(c) $\begin{cases} x = \sqrt{5} \cos \theta , \\ y = \sqrt{5} \sin \theta . \end{cases}$

17. (a) The path of P is a cycloid with parametric equations

$$\begin{cases} x = a(\theta - \sin \theta) , \\ y = a(1 - \cos \theta) . \end{cases}$$

We assume the following: $a = 12$ inches; the wheel rolls from left to right; x is measured in inches along the road to the right from the first contact point of P ; y is measured in inches above the road; θ is the angle of rotation measured clockwise from the 6 o'clock position to the position of P ; $\theta = \omega t$ where t is measured in seconds and $\omega = 3$ rps $= 6\pi$ radius per second. Our equations are:

$$\begin{cases} x = 12(6\pi t - \sin 6\pi t) , \\ y = 12(1 - \cos 6\pi t) . \end{cases}$$

(b) The path of Q is a curtate cycloid whose equations were derived in the solution to Challenge Exercise 2 on page 18.
The equations of the path of Q are

$$\begin{cases} x = 12(6\pi t) - 6 \sin (6\pi t) . \\ y = \quad 12 \quad - 6 \cos (6\pi t) . \end{cases}$$

Chapter 6

CURVE SKETCHING AND LOCUS PROBLEMS

This chapter explores in detail the relation between a curve and its analytic representation. We present some methods of curve-sketching which are probably new to the students, particularly the addition and multiplication of ordinates and the addition of radii. These relate the graphs of certain types of equations to the more familiar graphs of simpler equations.

We then discuss some geometric properties of the curve and see how they can be deduced from its analytic representations. We see how the choice of coordinate system casts its particular light on our analysis, and explore the advantages and disadvantages of each in a variety of situations. The geometric properties we consider are symmetry, extent, periodicity, intercepts, and asymptotes. The treatment is careful but not exhaustive, and students should be encouraged to see any open questions that we leave, and to try to supply some answers. This is the essence of research, and should be so presented.

We suggest some topics that may be explored as extensions of the content of this chapter: asymptotes which are oblique lines; asymptotes which are other curves; "phase displacements", which may be considered additions of abscissas; properties of families of curves; envelopes of families of curves; self-intersecting curves; extensions to three or more dimensions; applications of this content to physics, particularly to periodic phenomena such as radio broadcasting; the relations among period, frequency, velocity of propagation, and wave length; resonance and interference, both in sound and in light; the Doppler effect in sound and in light; and so on. Students are pleased to recognize the Doppler effect in the changing pitch of an automobile siren as it approaches, passes, and recedes from them. They are also pleased to observe the interference of light as they look through an almost closed space between thumb and forefinger.

The teacher is referred to any recently written text in physics, and particularly to the members of the science department in the school. The topics mentioned are suitable for joint investigation through experimental and theoretical approaches. Both students and teachers can benefit from a systematic investigation in depth of any of the topics mentioned, and the opportunity to check experiment with theory and vice versa.

The sine curve is particularly suited to exhibit such matters as boundedness and periodicity. The polar graph of $r = \sin\theta$ exhibits boundedness in that it is entirely contained in a circle of radius more than $\frac{1}{2}$. The periodicity is shown by the fact that as θ increases without limit, the point P will go endlessly around the circle as shown.

Exercises 6-2(a)

It is to be understood that when we ask for bounds for a graph we want the "best" bounds, that is, the most restrictive. Thus, for 1(a), $y = 2\sin x$, we certainly have bounds ± 10; "better" bounds are ± 5, but the "best" bounds are ± 2 as indicated below.

1. We use the fact that $0 \le \sin\theta \le 1$ and $0 \le \cos\theta \le 1$ for any θ.

   (a) $-2 \le y \le 2$ for any x.

   (b) $-1 \le y \le 1$ for any x.

   (c) $1 \le y \le 3$ for any x.

   (d) $-\frac{1}{2} \le y \le \frac{1}{2}$ for any x.

   (e) Since $0 \le \sin\theta \le 1$ for any θ, we have $0 \le 2\sin(3x + \frac{\pi}{2}) \le 2$, and $-2 \le y \le 6$.

   (f) We know $0 \le |0.6 \sin x| \le 0.6$, and $0 \le |0.8 \cos x| \le 0.8$ therefore we have bounds $0 \le y \le 1.4$; but we can do better, since the two terms in the sum, being related, do not reach their maximum (or minimum) values for the same value of x.

   Note that $(.6)^2 + (.8)^2 = 1$ therefore we may take $0.6 = \cos t$ and $0.8 = \sin t$, and write $y = \sin x \cos t + \cos x \sin t$, with t as above. Therefore $y = \sin(x + t)$ and we now have $-1 \le y \le 1$. These are the best bounds, and the solution to this exercise.

(g) $y = 2 \sin x + 3 \cos x = \sqrt{2^2 + 3^2}\left(\frac{2}{\sqrt{2^2+3^2}} \sin x + \frac{3}{\sqrt{2^2+3^2}} \cos x\right)$

$= \sqrt{2^2 + 3^2}$ (sin x cos t + cos x sin t) where $\cos t = \frac{2}{\sqrt{13}}$,

and so on.

Since $y = \sqrt{13} \sin(x + t)$ we have the solution,

$$-\sqrt{13} \le y \le \sqrt{13}$$

(h) $y = a \sin x + b \cos x = \sqrt{a^2 + b^2}\left(\frac{a}{\sqrt{a^2+b^2}} \sin x + \frac{b}{\sqrt{a^2+b^2}} \cos x\right)$

$= \sqrt{a^2 + b^2}$ (sin x cos t + cos x sin t) $= \sqrt{a^2 + b^2} \sin(x + t)$;

as in the previous solution. Therefore

$$-\sqrt{a^2 + b^2} \le y \le \sqrt{a^2 + b^2}$$

(i) $0 \le y \le 1$

(j) $y = \sin^2 x - \cos^2 x = -\cos 2x$. Therefore $-1 \le y \le 1$.

2. Bounds: $a - |b| \le y \le a + |b|$

Period: Since $\sin(cx + d) = \sin(cx + d + 2\pi n)$

$= \sin\left(c(x + \frac{2\pi n}{c}) + d\right)$, there will be no change in y if x is increased by $\frac{2\pi n}{c}$ for integral n . Therefore the period is $\frac{2\pi}{c}$.

6-2(b). Symmetry.

We deal only with point and line symmetry. The content of this section is essential to some important transformations of the plane, which will be dealt with in Chapter 10 and its supplement. Students should be cautioned against replacing the phrase "symmetric with respect to the x-axis", by the non-equivalent, "symmetric with the x-axis". Some authors use "wrt" to replace "with respect to". We usually confine the domain of θ thus: $0 \le \theta < 2\pi$, since the generalization beyond this domain is usually simple.

Exercises 6-2(b)

1. This question repeats Number 8 of Section 5-6, to whose solution you are referred.

2. An ellipse, or a rectangle which is not square; an equilateral triangle; a square.

3. A circle, a line, the plane, a half-plane.

4. Yes. In review exercise Number 17 at the end of this chapter we ask for the proof of a somewhat stronger statement, that symmetry with respect to both of two perpendicular lines requires symmetry with respect to their intersection.

5. No. Consider the graph of $xy = 1$, or $y = x^3$, or the letter S .

6. We summarize the results by tabulating for parts (a) , (b) , (c) , (d) , (e) , (h) , (i) , (j) , the answers to these questions: Is the graph symmetric with respect to the x-axis?; the y-axis?; origin?; the line $y = x$?; the line $y = -x$?

   (a) No, yes, no, no, no.
   (b) No, no, yes, no, no.
   (c) No, yes, no, no, no.
   (d) No, no, no, no, no.
   (e) No, no, no, yes, no.
   (f) This equation is equivalent to $(x + y)^2 + 2(x + y) + 1 = 2$ or $(x + y + 1)^2 = 2$, whose graph is the pair of lines $x + y + 1 \pm \sqrt{2} = 0$. These lines are parallel and are symmetric with respect to (1) the line midway between them: $x + y + 1 = 0$; (2) each point of this line, that is, each point $\{(x,y) : x = -t , y = t - 1 , \text{ for all } t\}$; and (3) each perpendicular to this line, that is each line of the family $x - y + k = 0$.
   (g) This equation is equivalent to $(x + y + 5)(x + y - 2) = 0$, whose graph is the pair of parallel lines: $x + y + 5 = 0$ and $x + y - 2 = 0$. They are symmetric with respect to (1) the line midway between them, $x + y + \frac{3}{2} = 0$; (2) each point of this line, that is, each point $\{(x,y) : x = -t , y = t - \frac{3}{2} , \text{ for all } t\}$; and (3) each perpendicular to this line, that is, to each line of the family $x - y + k = 0$.
   (h) No, no, no, yes, no.

(i) Yes, no, no, no, no.

(j) If n is even: yes, yes, yes, yes, yes;
if n is odd: no, no, no, yes, no.

For parts (k) - (t). We consider only symmetry with respect to the pole and any line through the pole. We present our answers in this order:
Is the graph symmetric with respect to the pole?
What lines through the pole are axes of symmetry for the graph?

(k) Yes; $\theta = 0$, $\theta = \frac{\pi}{2}$.

(ℓ) No; $\theta = 0$ (since the related polar equation is $r = -\sin^2 \theta$) , $\theta = \frac{\pi}{2}$.

(m) No; $\theta = \frac{\pi}{2}$. (This curve is an ovaloid through the points $(2,0)$, $(1,\frac{\pi}{2}),(2,\pi),(3,\frac{3\pi}{2})$.)

(n) No; $\theta = 0$. (This curve is a parabola.)

(o) No; $\theta = \frac{\pi}{2}$. (This curve is an ellipse. It has symmetry with respect to its center, the point $(\frac{3}{4},\frac{\pi}{2})$, and the lines along its axes. These lines are most easily represented in rectangular coordinates: $x = 0$, which has already been found, and $y = \frac{3}{4}$. This last result could be found by polar methods but will not be discussed further.)

(p) Yes; $\theta = 0$, $\theta = \frac{\pi}{2}$. (This locus is a pair of parallel lines and has, beside the axes of symmetry already mentioned, any line parallel to the polar axis, that is, any member of the family $r = \frac{a}{\sin \theta}$.

(q) Yes; $\theta = \frac{\pi}{4}$, $\theta = \frac{3\pi}{4}$. (This locus is a double loop in the first and third quadrants, crossing at the pole.)

(r) No; $\theta = \frac{\pi}{6}$, $\theta = \frac{5\pi}{6}$, $\theta = \frac{\pi}{2}$. (This locus is a three-leaved rosette, with loops out to $(2,\frac{\pi}{6})$, $(2,\frac{5\pi}{6})$, $(2,\frac{3\pi}{2})$.)

(s) No; $\theta = \frac{\pi}{2}$. (The graph is an ovaloid curve through the points $(3,0)$, $(1,\frac{\pi}{2})$, $(3,\pi)$, $(5,\frac{3\pi}{2})$.)

(t) No; $\theta = \frac{\pi}{2}$. (The graph is an ovaloid curve through the points $(a,0)$, $(a + b,\frac{\pi}{2})$, (a,π), $(a - b,\frac{3\pi}{2})$.)

Challenge Problems

1. Given a point P in space and a plane M which does not contain P. The symmetric image of P with respect to M is the point P' such that M is the perpendicular bisector of $\overline{PP'}$. The question of figure-reversal in a mirror can raise some interesting problems. The fact is that there is a top-bottom reversal, as is seen by the reflection of a mountain in the surface of a lake. We could easily see the top-bottom reversal in our persons if we stood on a mirror, or sat at a mirror-top desk. Our normal position of viewing establishes an unconscious vertical plane of reference, usually the perpendicular bisector of the segment joining our eyes. When we lie on our sides this plane is no longer vertical, and the reversal is now from top to bottom. You may grasp these ideas more clearly if you close one eye to help remove the unconscious vertical plane of reference and then consider various relative positions of the mirror, the eye, and the reflected object.

2. The problem is trivial if L is horizontal or vertical. Assume that it is neither. L is the ⊥ bisector of $\overline{P_1P_2}$, therefore the midpoint of $\overline{P_1P_2}$ must be on L, therefore $a\left(\frac{x_1 + x_2}{2}\right) + b\left(\frac{y_1 + y_2}{2}\right) + c = 0$, or $ax_1 + ax_2 + by_1 + by_2 + 2c = 0$. $\overleftrightarrow{P_1 P_2} \perp L$, therefore $\frac{y_2 - y_1}{x_2 - x_1} = \frac{b}{a}$, thus, $bx_1 - bx_2 - ay_1 + ay_2 = 0$. We solve these two equations for x_2 and y_2, and find

$$P_2 = \left(-\frac{(a^2 - b^2)x_1 + 2ab\, y_1 + 2ac}{a^2 + b^2},\ -\frac{2ab\, x_1 + (b^2 - a^2)y_1 + 2bc}{a^2 + b^2}\right)$$

6-2(c). Extent.

The discussions of the examples in the text are done in sufficient detail to meet the requirements of a text at this level. The ideas of this section are a good foundation for the topic of continuity which is so significant in the calculus. We do not discuss functions whose graphs have serious discontinuities; nor the "pathological" curves of higher mathematics. However, it is salutary for the class to discuss the graph of, say, $y = (-1)^x$ which is totally discontinuous and consists of an infinite number of the points of the lines $y = 1$ and $y = -1$.

In this chapter (Page 214), the term "asymptote" has been used with reference to a line to which the points on a graph approach more and more closely, but which contain no points of the graph. This is always true of the vertical asymptotes, i. e. the y-axis or lines parallel to the y-axis. In the second example (see Figure 6 - 9), we note that the x-axis is crossed by the curve at (0,0) but acts as an asymptote for the points of the graph where $x > 1$ and $x < -3$. In common practice such a line is also referred to as a horizontal asymptote. However, it can be proved that such horizontal asymptote may have only a finite number of points in common with the curve.

It is also important to note that it is possible to have asymptotes which are not lines. For example, the parabola $y = x^2$ acts as an asymptote to the curve $y = x^2 + \frac{1}{x}$. You may like to assign this to your better students after discussing the graph of $y = x + \frac{1}{x}$ on page 220.

6-3. Conditions and Graphs (Rectangular Coordinates).

We have taken a good deal of time and space to show how to sketch certain graphs which are related to familiar graphs. Students soon "catch on" and quickly develop a fine competence in this part of their work, often reporting later that this was the most useful part of the course in later applications.

We suggest again a dynamic approach to graphing. Typical questions are, "How could we change the equation to raise the graph 2 units?" , "What happens to the graph if we reverse these signs?" , etc. As in all exercises, the more, the better, but please do not assign all the exercises of Exercise 6-3. You may, of course, use some of them for test items.

Students are always interested in applications of these ideas that come within their immediate experience. You should point out that the graph of the equation $y = a + b \sin cx$ is a simplified version of broadcast waves that are received by their radio and television sets. An increase in a has the effect of raising the "bias". Roughly this is what is done on the TV set when we increase the "brightness". An increase in b has the effect of increasing the amplitude. On the TV screen the lights would get lighter and the darks would get darker. This is what is done when we increase the "contrast".

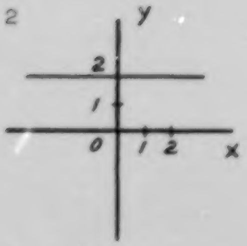
The equation $y = b \sin cx$ also represents roughly the motion of a point on a vibrating string. When we strike a piano key lightly, then heavily we increase the loudness but not the pitch. This situation would correspond in the equation, to increasing b , but keeping c constant.

When we strike two piano keys evenly we have the same loudness but different pitch. This would correspond in the equation to keeping b constant but changing c .

The relationships among mathematics, physics, and music were investigated by the great Greek mathematicians. We leave for individual investigation the extension and development of these ideas to include harmony, resonance, interference, beats, etc., all of which are referred to in any current book on physics.

Exercises 6-3

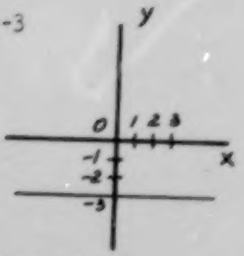
1. $y = 2$



y-intercept: 2

x-intercept: none

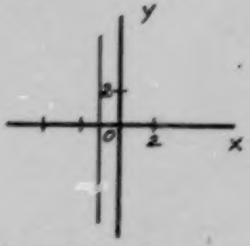
2. $y = -3$



y-intercept: -3

x-intercept: none

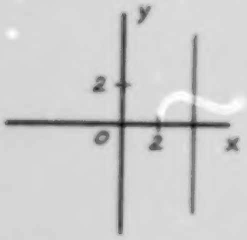
3. $x = -1$



y-intercept: none

x-intercept: -1

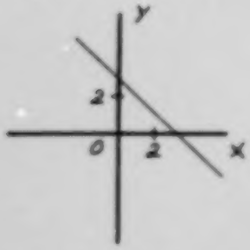
4. $x = 4$



y-intercept: none

x-intercept: 4

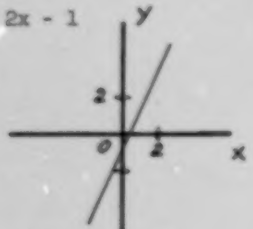
5. $y = -x + 3$



y-intercept: 3

x-intercept: 3

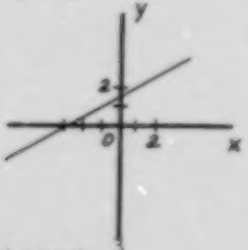
6. $y = 2x - 1$



y-intercept: -1

x-intercept: $\frac{1}{2}$

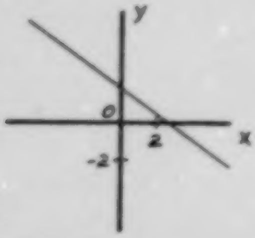
7. $x - 2y + 3 = 0$



y-intercept: $\frac{3}{2}$

x-intercept: -3

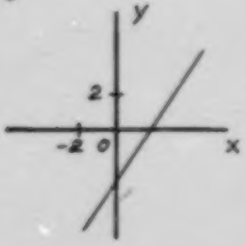
8. $2x + 3y - 5 = 0$



y-intercept: $\frac{5}{3}$

x-intercept: $\frac{5}{2}$

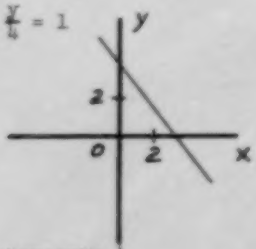
9. $\frac{x}{2} - \frac{y}{3} = 1$



y-intercept: -3

x-intercept: 2

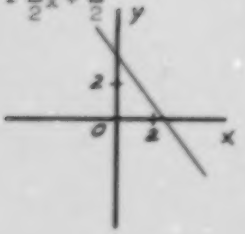
10. $\frac{x}{3} + \frac{y}{4} = 1$



y-intercept: 4

x-intercept: 3

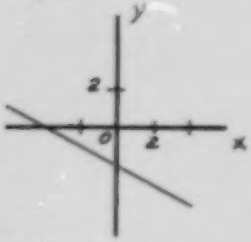
11. $y = -\frac{3}{2}x + \frac{7}{2}$



y-intercept: $\frac{7}{2}$

x-intercept: $\frac{7}{3}$

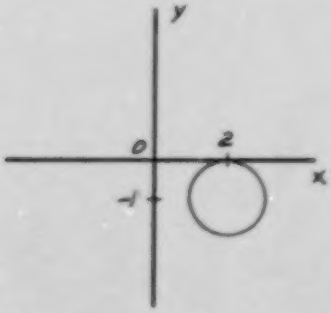
12. $y = -\frac{1}{2}x - 2$



y-intercept: -2

x-intercept: -4

13. $(x - 2)^2 + (y + 1)^2 = 1$



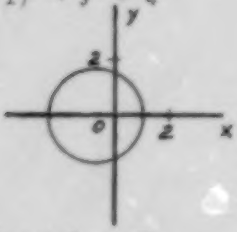
y-intercept: none

x-intercept: 2

center: $(2,-1)$

bounded

14. $(x + 1)^2 + y^2 = 4$



y-intercept: $\pm\sqrt{3}$

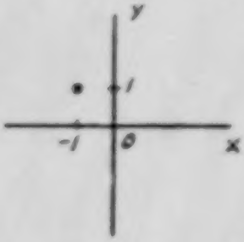
x-intercepts: -3,1

center: (-1,0)

bounded

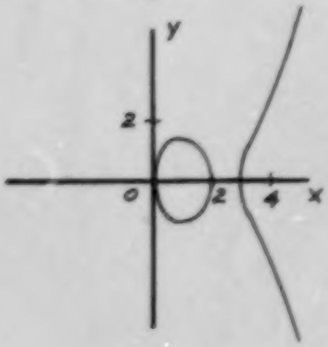
15. $(x + 1)^2 + (y - 1)^2 = 0$

which is $x = -1$ and $y = 1$



The locus is a single point.

16. $y^2 = x(x - 2)(x - 3)$



y-intercept: 0

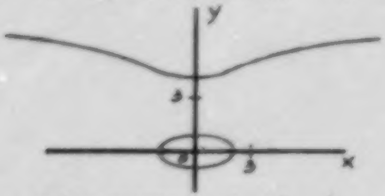
x-intercepts: 0 , 2 , 3

curve is not connected

symmetric in x-axis

no asymptotes

17. $x^2 = (y + 1)(y - 1)(y - 4)$



y-intercepts: -1 , 1 , 4

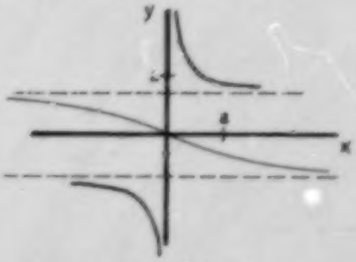
x-intercepts: -2 , 2

curve is not connected

symmetric in y-axis

no asymptotes

18. $xy^2 - 2y - x = 0$



y-intercepts: 0

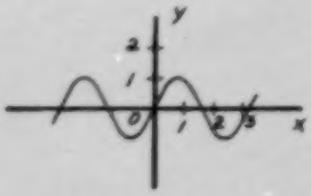
x-intercept: 0

not connected

symmetric in point (0,0)

asymptotes: $y = 1$, $y = -1$, $x = 0$

19. $y = \sin 2x$

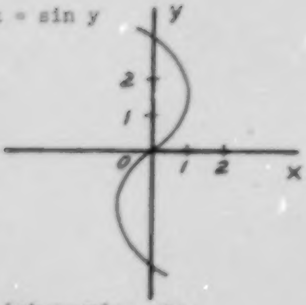


y-intercept: 0

x-intercepts: $n\frac{\pi}{2}$

$-1 \leq y \leq 1$

20. $x = \sin y$

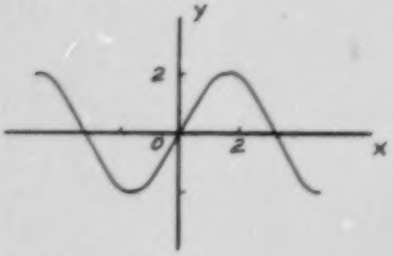


y-intercepts: $n\pi$

x-intercept: 0

$-1 \leq x \leq 1$

21. $y = 2 \sin x$

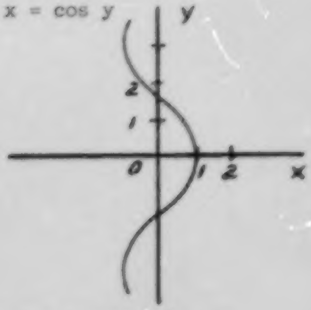


y-intercept: 0

x-intercepts: $n\pi$

$-2 \leq y \leq 2$

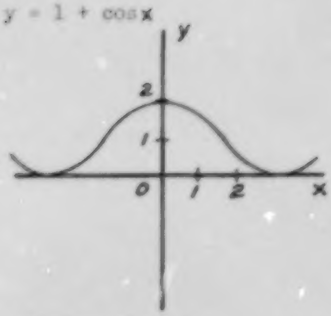
22. $x = \cos y$



y-intercepts: $\frac{\pi}{2} + n\pi$

x-intercept: 1

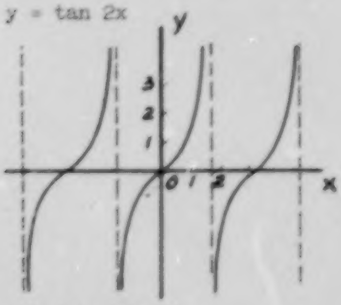
23. $y = 1 + \cos x$



y-intercept: 2

x-intercepts: $(2n + 1)\pi$

24. $y = \tan 2x$

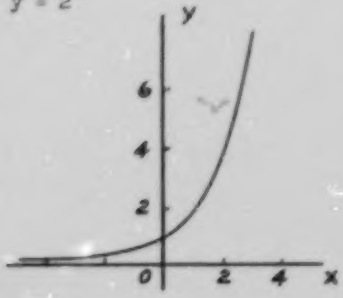


y-intercept: 0

x-intercepts: $n\pi$

asymptotes: $x = \frac{2n + 1}{2}\pi$

25. $y = 2^x$

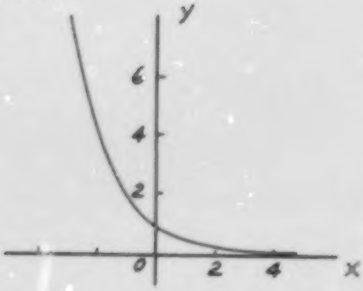


y-intercept: 1

x-intercept: none

asymptote: $y = 0$

26. $y = 2^{-x}$

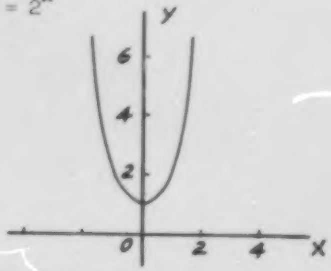


y-intercept: 1

x-intercept: none

asymptote: $y = 0$

27. $y = 2^{x^2}$

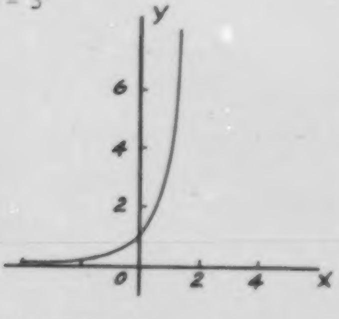


y-intercept: 1

x-intercept: none

symmetric in y-axis

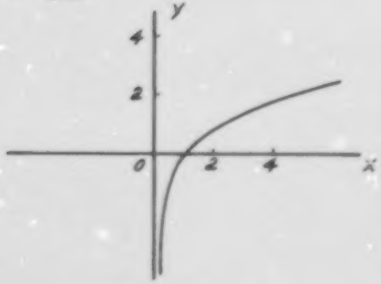
28. $y = 3^{x^3}$



y-intercept: 1

x-intercept: none

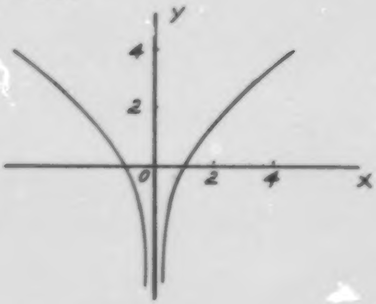
29. $y = \ell nx$



y-intercept: none

x-intercept: 1

30. $y = \ell nx^2$

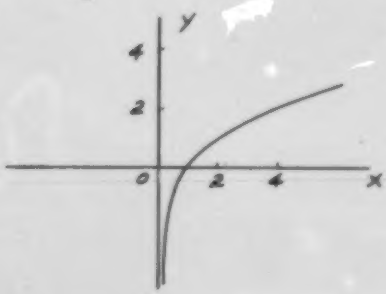


y-intercept: none

x-intercepts: 1 , -1

symmetric wrt y-axis

31. $y = \log_2 x$

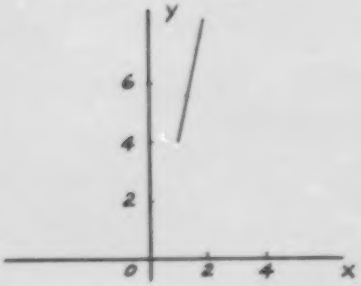


y-intercept: none

x-intercept: 1

asymptote: $x = 0$

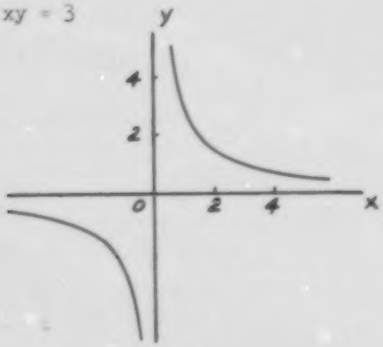
32. $y = 5x - 1$ and $x \geq 1$



no intercepts

no points in left or below $(1,4)$

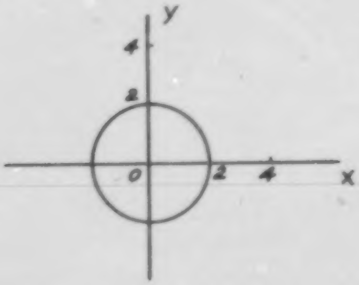
33. $xy = 3$



no intercepts

asymptotes: $x = 0$, $y = 0$

34. $x^2 + y^2 = 2^2$



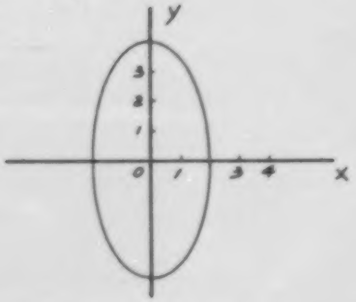
y-intercepts: 2 , -2

x-intercepts: 2 , -2

center: $(0,0)$

symmetric wrt any line through 0 .

35. $4x^2 + y^2 = 16$



y-intercepts: 4 , -4

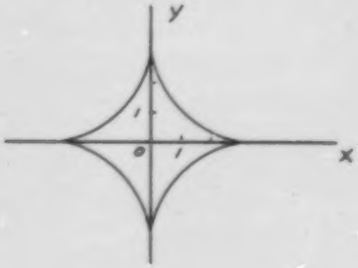
x-intercepts: 2 , -2

symmetric in both axes

$-4 \leq y \leq 4 \quad -2 \leq x \leq 2$

36. $x = 3 \cos^3 \theta \qquad y = 3 \sin^3 \theta$

$x^{2/3} + y^{2/3} = 9^{2/3}$



y-intercepts: 3 , -3

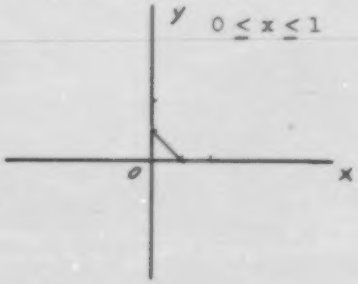
x-intercepts: 3 , -3

symmetric wrt both axes

tangent to axes at corner points

37. $x + y = 1 \qquad 0 \leq y \leq 1$

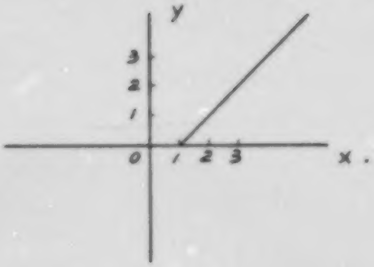
$0 \leq x \leq 1$



y-intercept: 1

x-intercept: 1

38. $y = x - 1$ and $x \geq 1$ and $y \geq 0$

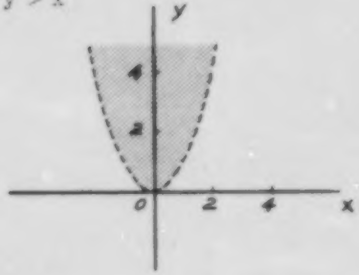


y-intercept: none

x-intercept: 1

restricted to first quadrant

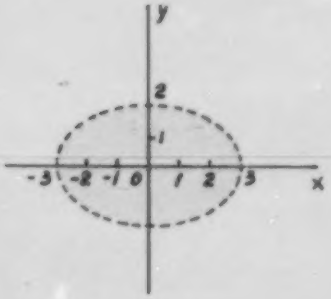
39. $y > x^2$



symmetric wrt y-axis

unbounded

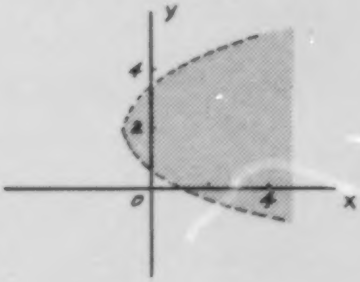
40. $\frac{x^2}{9} + \frac{y^2}{4} < 1$



bounded

symmetric wrt both axes

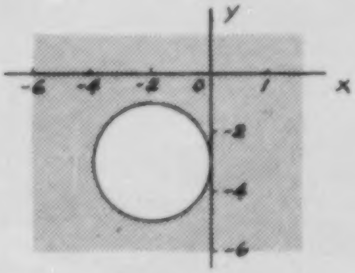
41. $(y - 2)^2 < 2(x + 1)$



unbounded region

symmetric wrt $y = 2$

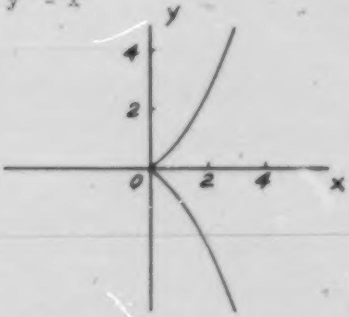
42. $(x + 2)^2 + (y + 3)^2 \geq 4$



unbounded region

center at $(-2,-3)$

43. $y^2 = x^3$

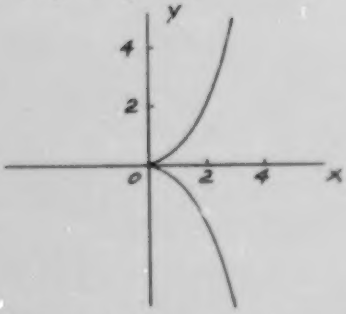


y-intercept: 0

x-intercept: 0

symmetric wrt x-axis

44. $x^3 + xy^2 - 4y^2 = 0$

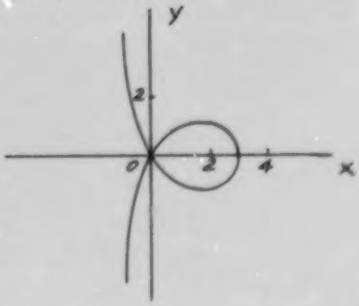


y-intercept: 0

x-intercept: 0

symmetric wrt x-axis

45. $x^3 + xy^2 - 3x^2 + y^2 = 0$

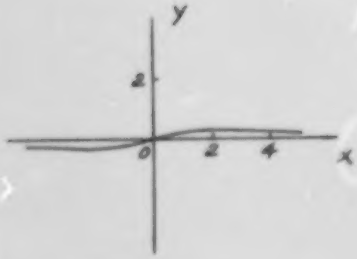


y-intercept: 0

x-intercepts: 0 , 3

asymptote: $x = -1$

46. $x^2y + 4y - x = 0$



y-intercept: 0

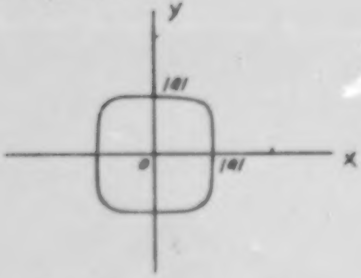
x-intercept: 0

asymptote: $y = 0$

symmetric wrt origin

$\frac{-1}{4} \le y \le \frac{1}{4}$

47. $x^4 + y^4 = a^4$



y-intercepts: $|a|$, $-|a|$

x-intercepts: $|a|$, $-|a|$

symmetric wrt both axes

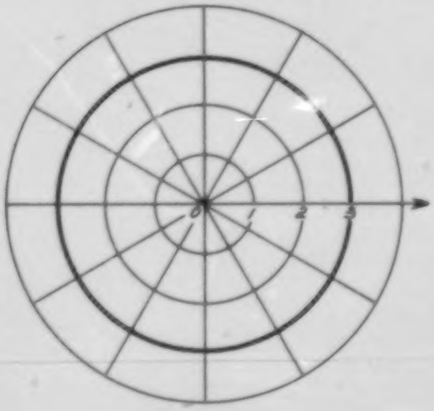
6-4. Graphs and Conditions (Polar Coordinates).

The use of an auxiliary graph in rectangular coordinates, as shown in Example 3 is probably new to the class. It is a useful technique and should be practised in a few exercises until it is understood and becomes a familiar tool. The same may be said for the technique of addition of radii, shown in the same example. We may think of this last technique in a dynamic way, considering the radius, r , as changing, or modulating, as θ changes. Thus, in Figure 6-29(a), as the ray $\overleftrightarrow{OP}$ rotates counterclockwise the Q-points along that ray also move counterclockwise, but have an extra radial motion, the modulation of the radii. The class might discuss the graphs of $1 + 3 \sin \theta$; $1 + b \sin \theta$; $2 + \sin \theta$; $a + \sin \theta$; and finally $a + b \sin \theta$, for changing values of a and b .

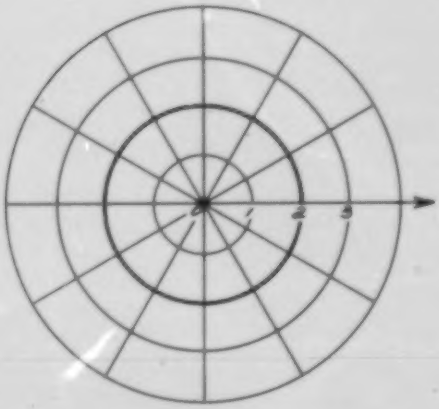
The special ambiguity in the polar coordinates of the pole is an extra ingredient to consider in discussing the intersections of polar graphs. The situation has a geographic analog which students find interesting. If you are at the north pole, which direction is south? The answer is more semantic than factual. If by "south" we mean directly toward the south end of the earth's axis then the answer is: straight down along that axis. If by "south" we mean an available direction of travel along the earth's surface, then the answer is: any direction. If "north" and "south" mean "directly to the ends of the earth's axis", then an object dropped to the surface from a point above the "north pole" will travel simultaneously both north and south!

Exercises 6-4

1. $r = 3$, $r = -3$
Circle: center O , radius 3
$x^2 + y^2 = 9$



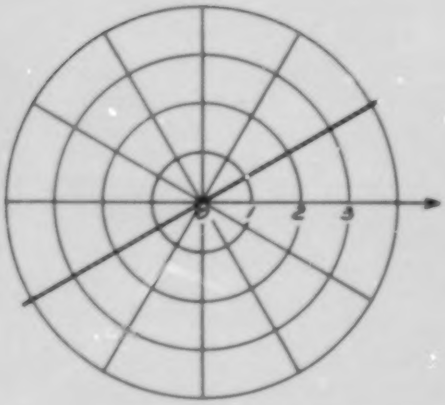
2. $r = -2$, $r = 2$
Circle: center O , radius 2
$x^2 + y^2 = 4$



3. $\theta = \frac{\pi}{6}$, $\theta = \frac{7\pi}{6}$

Line through 0

$y = \frac{\sqrt{3}}{3}x$

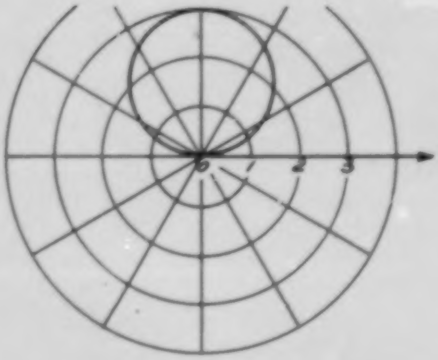


5. $r = 3 \sin \theta$, related equation the same

Circle: center $(\frac{3}{2}, \frac{\pi}{2})$, radius $\frac{3}{2}$

This circle is described twice as the radius vector rotates through 2π.

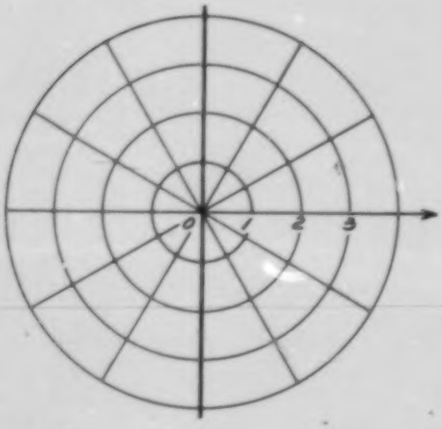
$x^2 + (y - \frac{3}{2})^2 = \frac{9}{4}$



4. $\theta = -\frac{3\pi}{2}$, $\theta = -\frac{\pi}{2}$

Line through 1

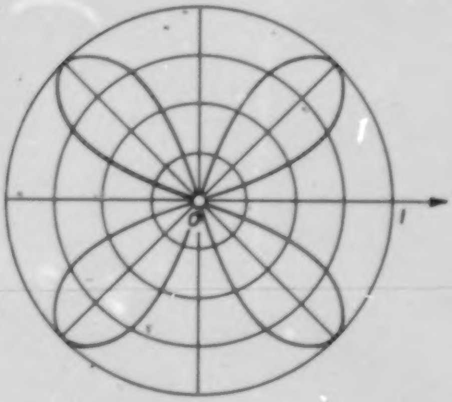
$x = 0$



6. $r = \sin 2\theta$, $r = -\sin 2\theta$

The graph is a four-leaved rose.

$(x^2 + y^2)^{3/2} = 2xy$

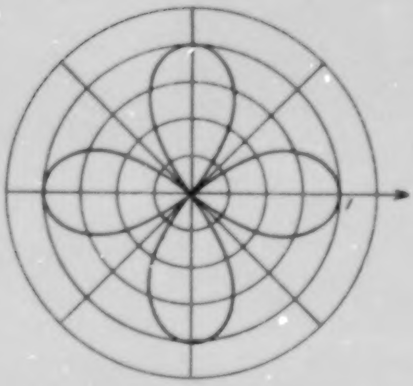


7. $r = \cos 2\theta$, $r = -\cos 2\theta$

Four leafed rose

Symmetric with respect to origin and lines $\theta = 0$, $\theta = \frac{\pi}{4}$, $\theta = \frac{\pi}{2}$

$(x^2 + y^2)^3 = (x^2 - y^2)^2$

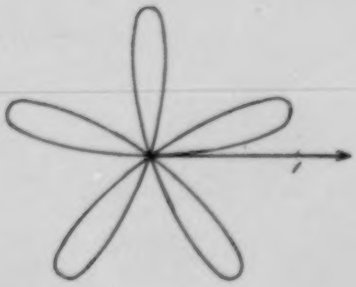


8. $r = \sin 5\theta$, related equation the same

Five leafed rose

Symmetric with respect to origin and lines $\theta = \frac{n\pi}{10}$, $n = 1,3,5,7,9$.

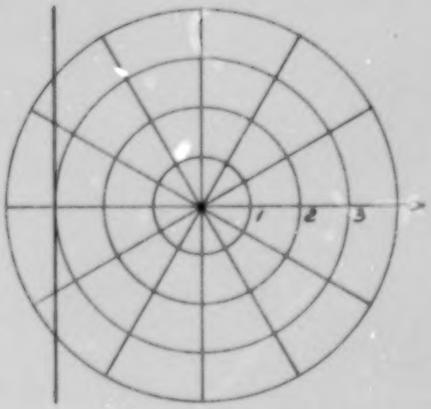
$(x^2 + y^2)^{3/2} = 5x^4y - 10x^2y^3 + y^5$



9. $r \cos \theta = -3$, related equation the same

Straight line

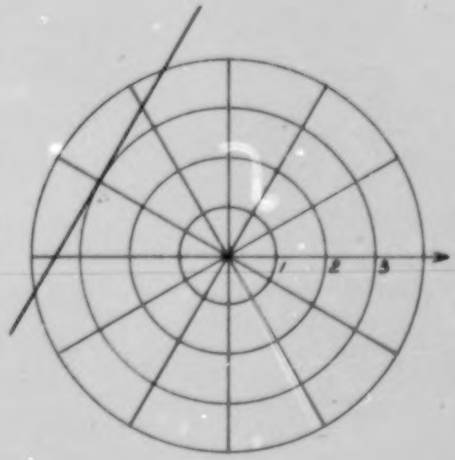
$x = -3$



10. $r \cos (\theta - 150^\circ) = 3$, related equation the same

Straight line

$y = \sqrt{3}x + 6$

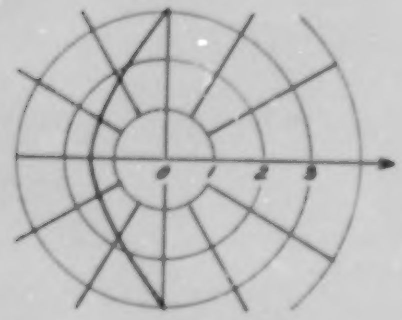


11. $r = \frac{3}{1 - \cos\theta}$, $r = \frac{-3}{1 + \cos\theta}$

Parabola: focus 0 , directrix
$x = -3$

Unbounded. Symmetric with respect to polar axis

$y^2 = 6x + 9$



12. $r = \frac{9}{4 - 5\cos\theta}$, $r = \frac{-9}{4 + 5\cos\theta}$

Hyperbola with $e = \frac{5}{4}$

foci at 0 and $(10, \pi)$

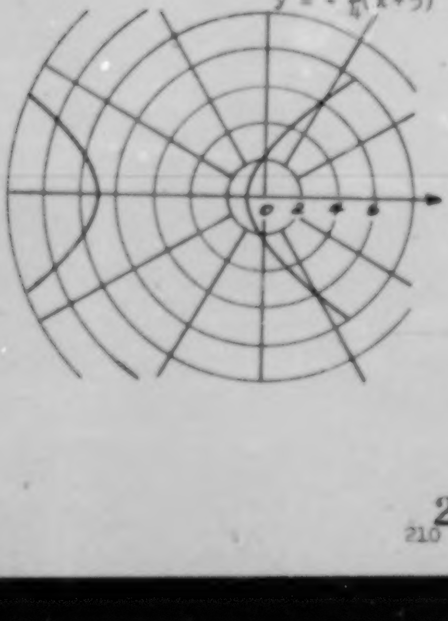
Center at $(5, \pi)$. Unbounded.

Symmetric with respect to center, line $x = -5$, and polar axis.

$9(x + 5)^2 - 16(y)^2 = 144$

The asymptotes are: $y = \frac{3}{4}(x + 5)$

$y = -\frac{3}{4}(x + 5)$

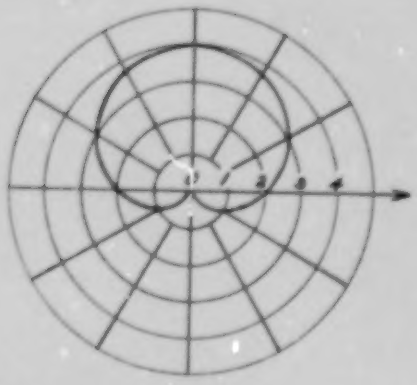


13. $r = 2(1 + \sin\theta)$, $r = 2(\sin\theta - 1)$

Cardiod: Bounded

Symmetric wrt $\theta = 90^\circ$

$(x^2 + y^2)^2 - 4x^2(1 + y) - 4y^3 = 0$



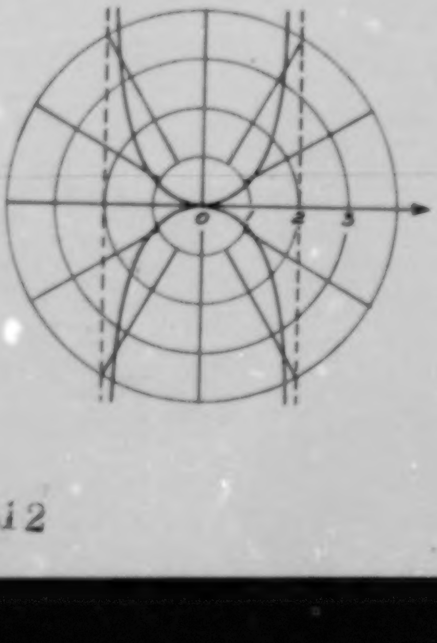
14. $r = 2\tan\theta$, $r = -2\tan\theta$

"Kappa Curve" so called because of its resemblance to the Greek letter kappa.

Unbounded. Symmetric wrt origin, $\theta = 0$, $\theta = 90^\circ$.

Vertical asymptotes $x = \pm 2$

$x^4 = y^2(4 - x^2)$

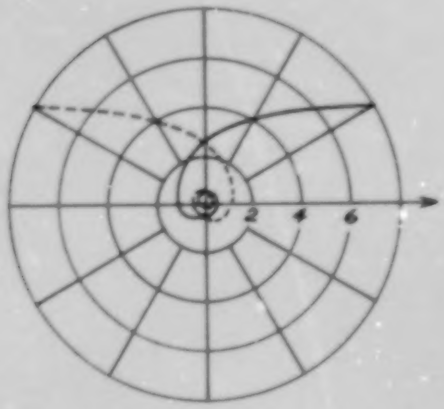


15. $r = \frac{4}{\theta}$, $r = \frac{-4}{\theta + \pi}$

Spiral. Unbounded.

(Solid line corresponds to positive r)

Not defined at $\theta = 0$ or $r = 0$

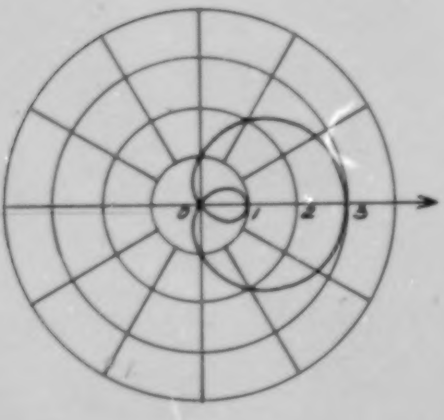


16. $r = 2 \cos \theta - 1$, $r = 2 \cos \theta + 1$

Limacon. Bounded.

Symmetric wrt $\theta = 0$

$(x^2 + y^2)^2 - 4x(x^2 + y^2) + 3x^2 - y^2 = 0$

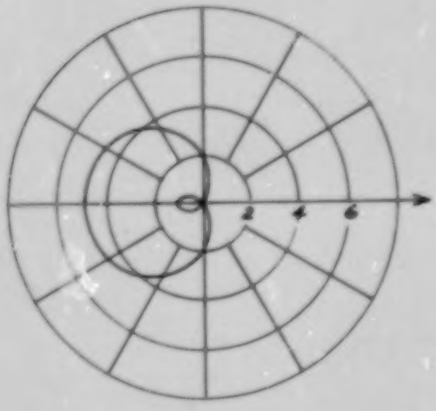


17. $r = 2 - 3 \cos \theta$, $r = -2 - 3 \cos \theta$

Limacon. Bounded.

Symmetric wrt $\theta = 0$

$(x^2 + y^2)^2 + 6x(x^2 + y^2) + 5x^2 - 4y^2 = 0$

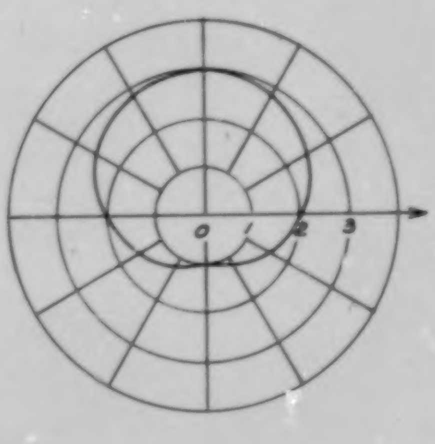


18. $r = 2 + \sin \theta$, $r = \sin \theta - 2$

Cardiod. Bounded.

Symmetric wrt $\theta = 90^\circ$

$(x^2 + y^2)^2 - 2y(x^2 + y^2) - y^2 - 2x^2 = 0$



19. $r^2 = \cos 2\theta$, related equation the same

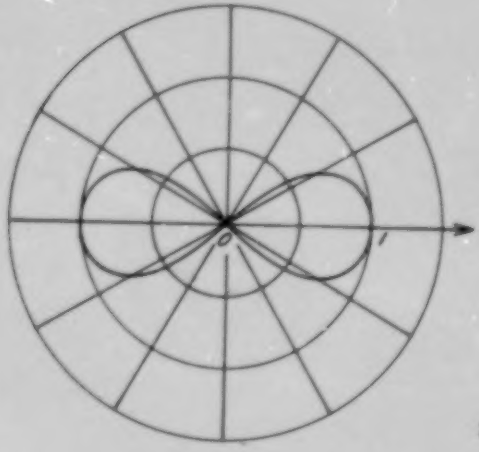
Two leafed rose

Symmetric wrt $\theta = 0$, $\theta = 90^\circ$.

Bounded, restricted to segments.

$-45^\circ \leq \theta \leq 45^\circ$, $135^\circ \leq \theta \leq 225^\circ$

$(x^2 + y^2)^2 = x^2 - y^2$



20. $r^2 = 4 \sin 2\theta$, related equation the same

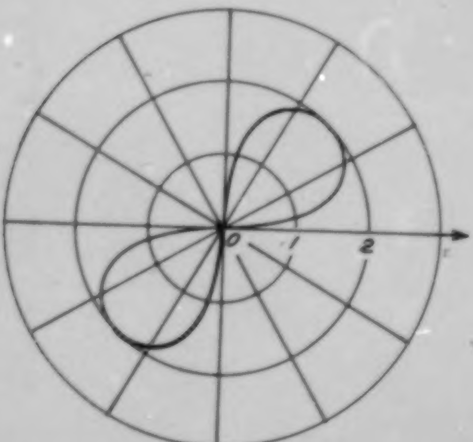
Two leafed rose

Symmetric wrt $\theta = 45^\circ$ and 0

Bounded, restricted to

$0 \leq \theta \leq 90^\circ$, $180^\circ \leq \theta \leq 270^\circ$

$(x^2 + y^2)^2 = 8xy$

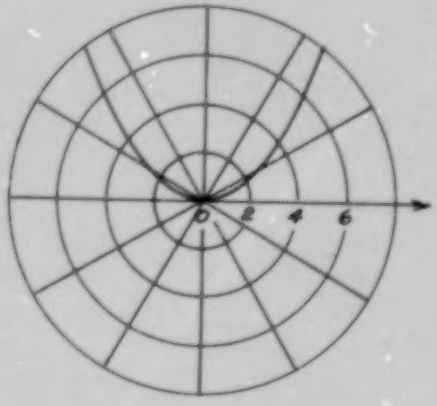


21. $r = 4 \tan \theta \sec \theta$, related equation the same

Parabola. Unbounded.

Symmetric wrt $\theta = 90^\circ$

$y = \frac{1}{4}x^2$

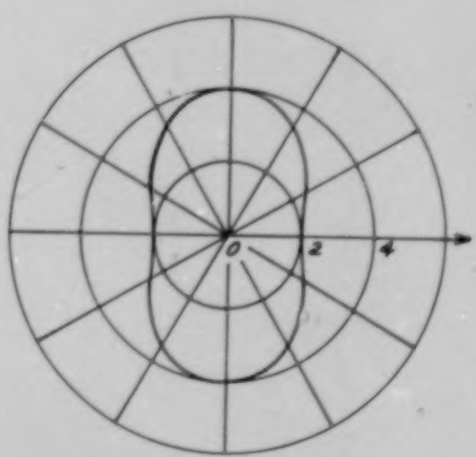


22. $r = 2(1 + \sin^2 \theta)$, $r = -2(1 + \sin^2\theta)$

Bounded.

Symmetric wrt $\theta = 0$, $\theta = 90^\circ$

$(x^2 + y^2)^3 = 4(x^2 + y^2 + y)^2$

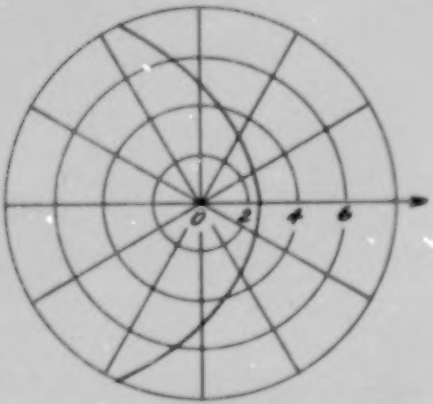


23. $r = \frac{5}{1 + \cos\theta}$, $r = \frac{5}{\cos\theta - 1}$

Parabola. Unbounded.

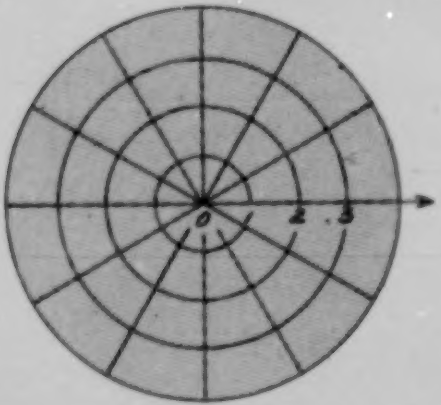
Symmetric wrt $\theta = 0$

$x = -0.1y^2 + 2.5$



24. $r \leq 2$, $r \geq 2$

The whole plane; every point (r,θ) in the plane may be expressed with negative r .

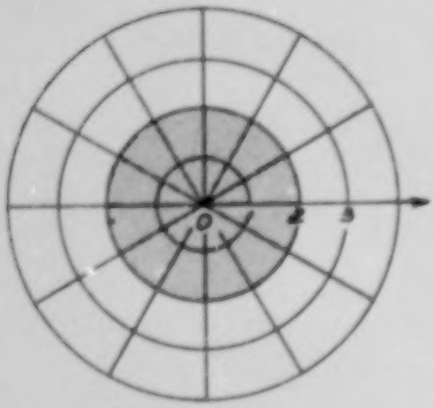


25. $|r| \leq 2$, related equation the same

Disk, boundary included.

Bounded. Symmetric wrt 0 .

$x^2 + y^2 \leq 4$

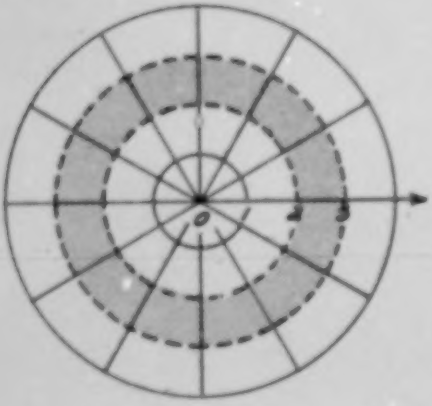


26. $2 < r < 3$, $-3 < r < -2$

Annulus, boundary not included

Bounded. Symmetric wrt 0 .

$4 < x^2 + y^2 < 9$



27. $0 \le \theta \le \frac{\pi}{4}$, $\pi \le \theta \le \frac{5\pi}{4}$

Unbounded. Symmetric

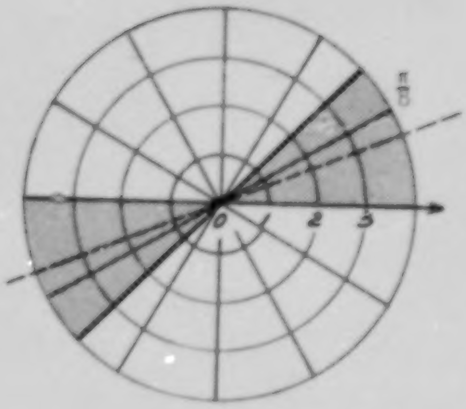
wrt O and line $\theta = \frac{\pi}{8}$

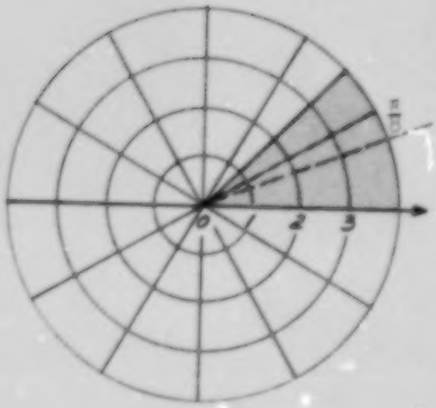
28. $0 \le \theta \le \frac{\pi}{4}$ and $r \ge 0$,

$\pi \le \theta \le \frac{5\pi}{4}$, and $r \le 0$.

Unbounded. Symmetric

wrt line $\theta = \frac{\pi}{8}$





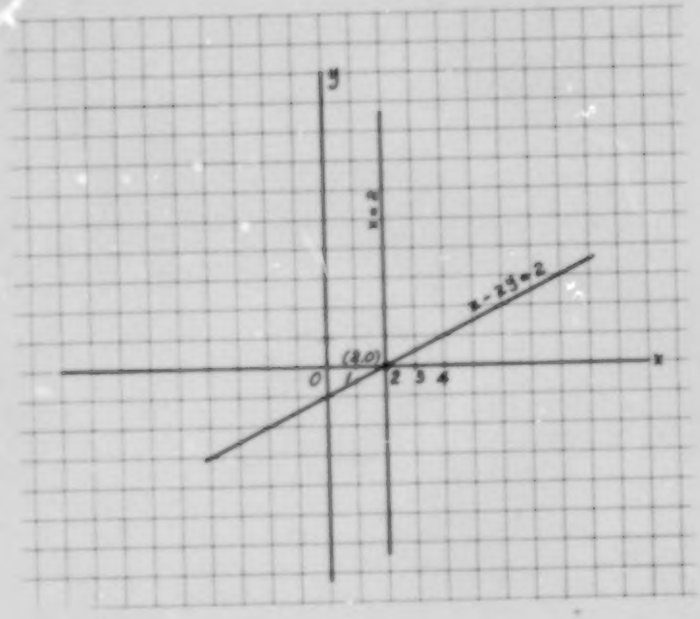
6-5. Intersections of Graphs (Rectangular Coordinates).

This topic has been met in earlier courses and is here treated with a little more generality. The method of linear combinations of functions is used briefly here and more thoroughly in Section 6-7. The exercises are limited to linear and quadratic equations only and present no special difficulties. Higher degree equations have more complicated graphs and present much more difficulties when we consider their intersections.

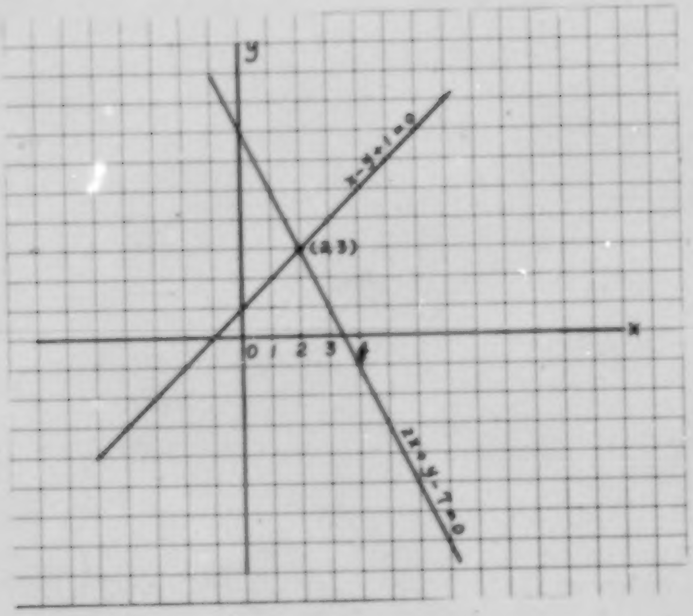
The order, n , of a curve is the maximum number of points of intersection that it may have with a straight line. Students may enjoy discussing the following questions about the orders of curves: What is the relation between the order of a curve and the degree of an equation of it? (Note that we say "an equation", because we have already seen that a curve may have more than one equation.) What is the maximum number of intersections between two curves of orders m and n ? Discuss the order of a closed curve, a self-intersecting curve.

Exercises 6-5

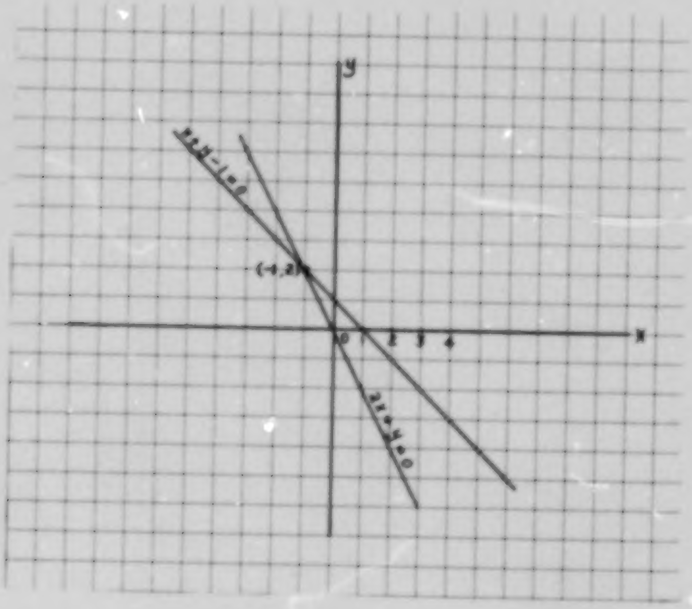
1. The point of intersection is (2,0).



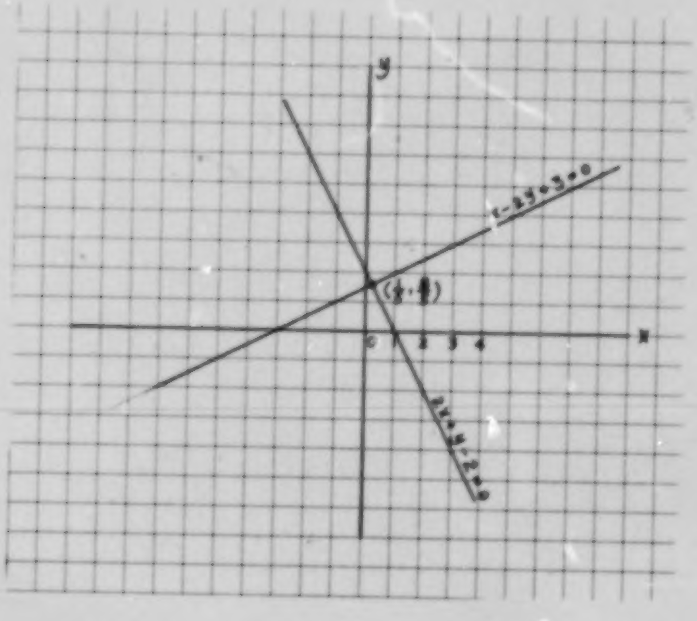
2. The point of intersection is (2,3).



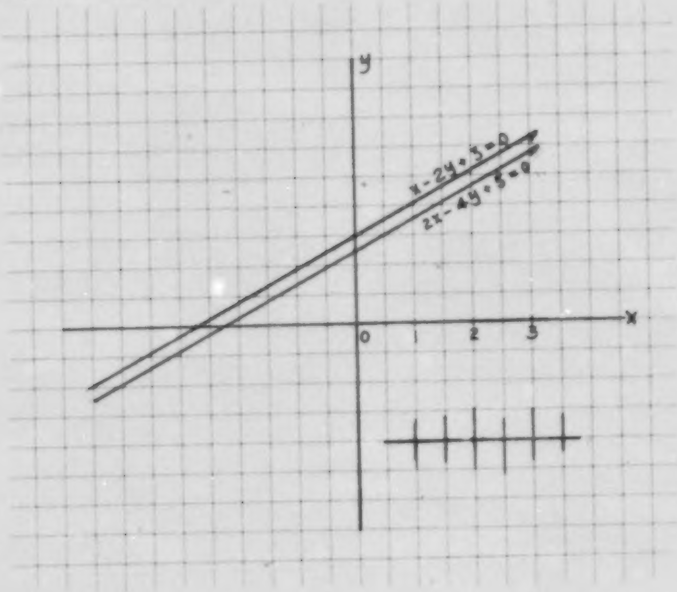
3. The point of intersection is (-1,2).



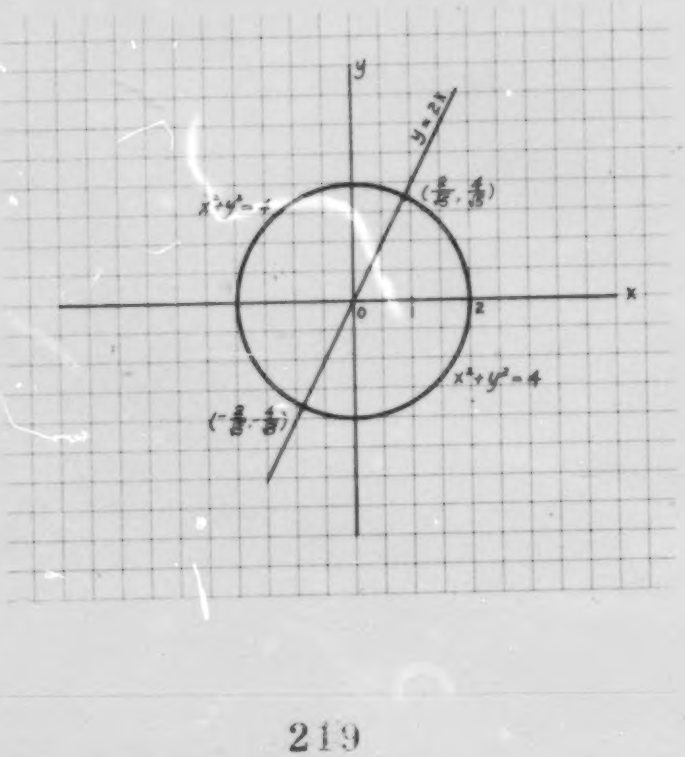
4. The point of intersection is $(\frac{1}{5}, \frac{8}{5})$.



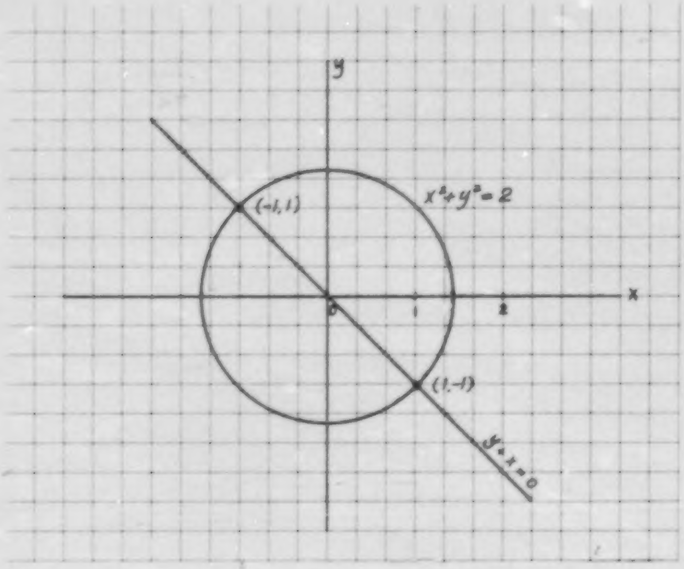
5. The intersection is the null set.



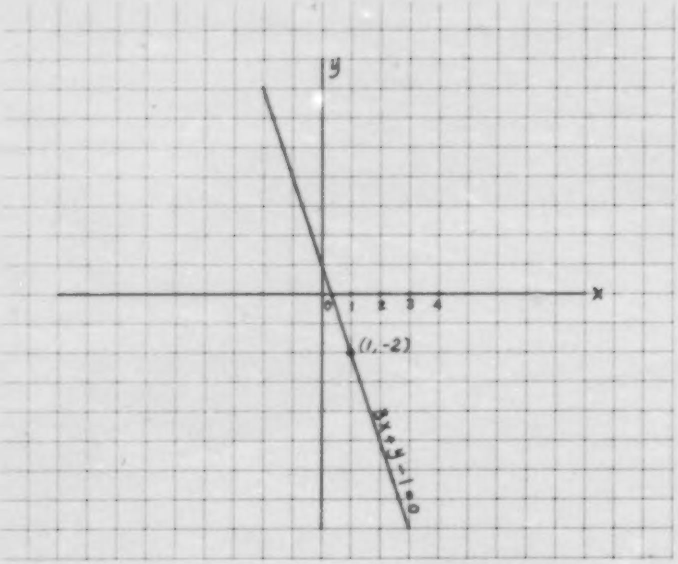
6. The points of intersection are $(\frac{2}{\sqrt{5}}, \frac{4}{\sqrt{5}})$ and $(-\frac{2}{\sqrt{5}}, -\frac{4}{\sqrt{5}})$



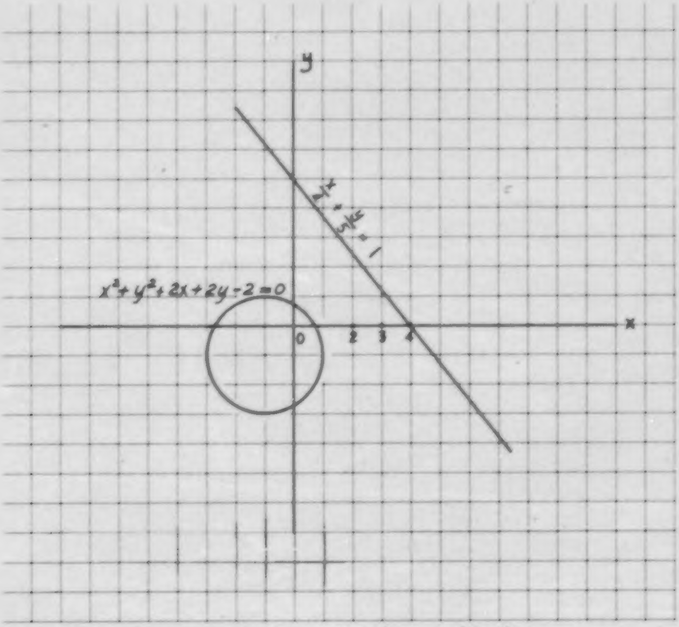
7. The points of intersection are (1, -1) and (-1, 1).



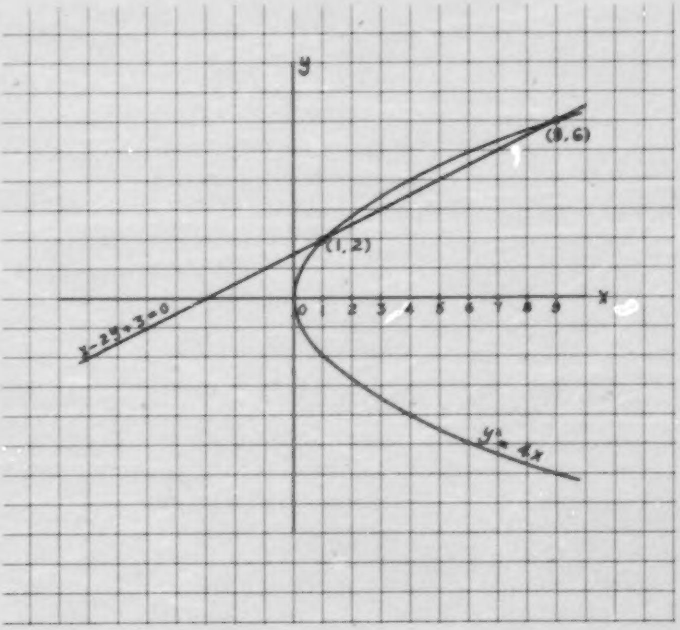
8. The point of intersection is (1, -2).



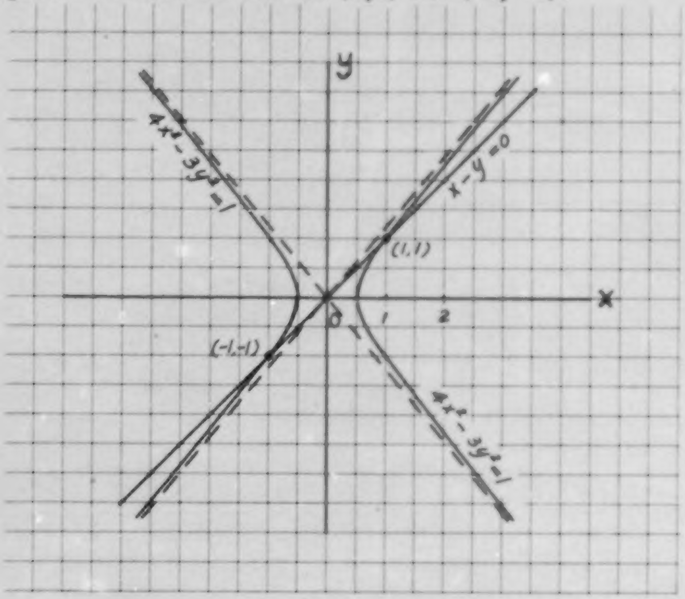
9. The intersection is the null set.


y
$\frac{x}{4} + \frac{y}{3} = 1$
$x^2+y^2+2x+2y-2=0$
0
2
3
4
x

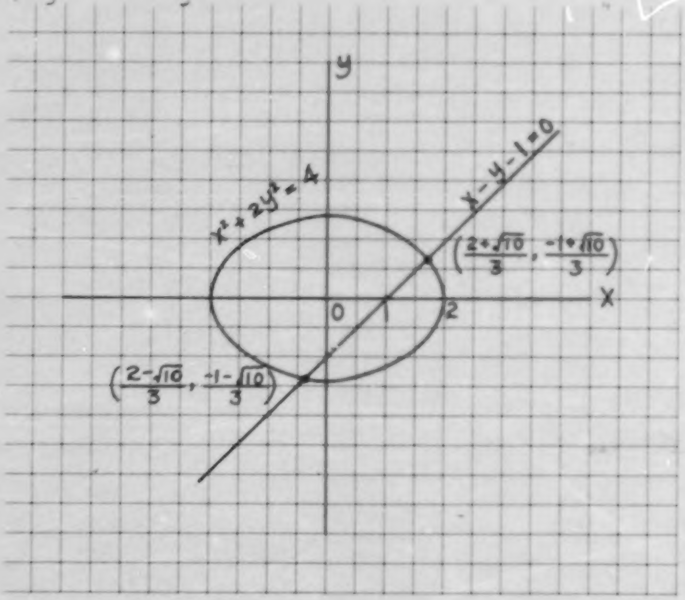
10. The points of intersection are (1,2) and (9,6).


y
(9,6)
(1,2)
0
1
2
3
4
5
6
7
8
9
x
$x-2y+3=0$
$y^2=4x$

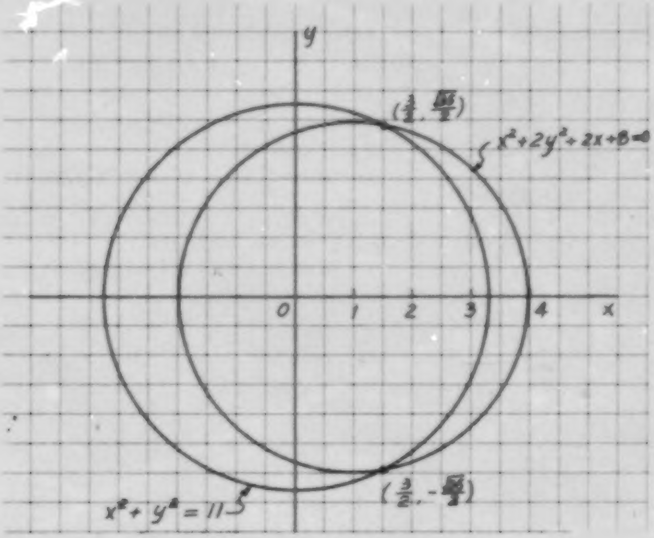
11. The points of intersection are (1,1) and (-1, -1).



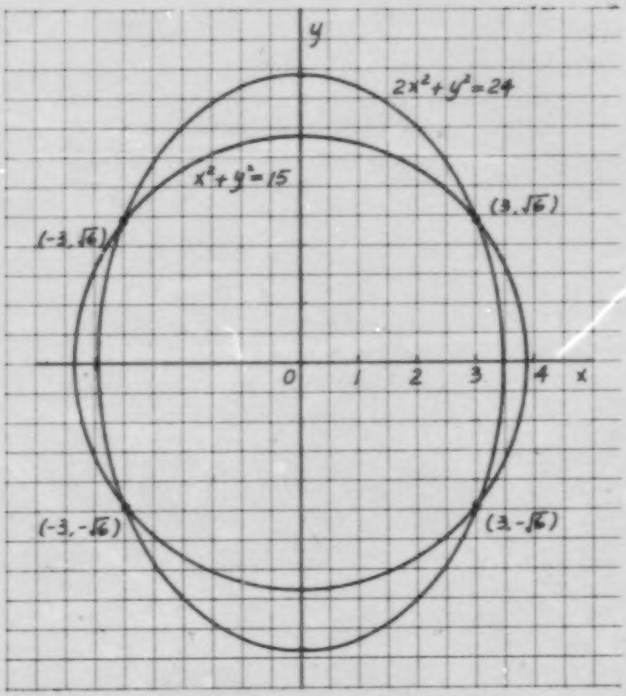
12. The points of intersection are $\left(\frac{2 + \sqrt{10}}{3}, \frac{-1 + \sqrt{10}}{3}\right)$ and $\left(\frac{2 - \sqrt{10}}{3}, \frac{-1 - \sqrt{10}}{3}\right)$



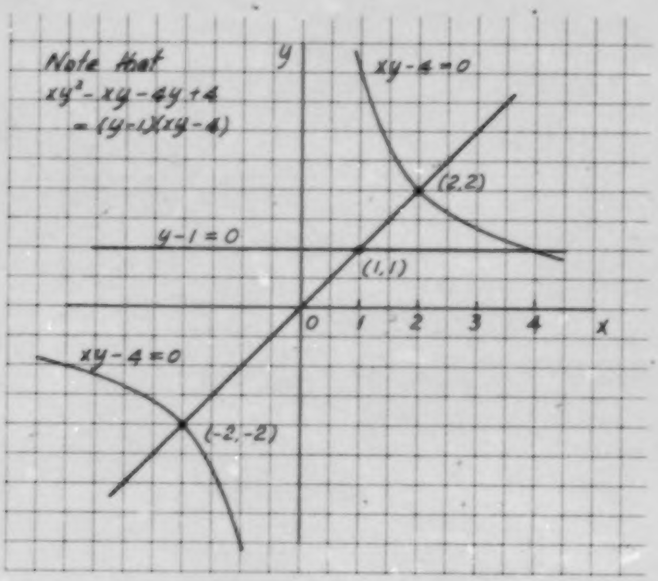
13. The points of intersection are $(\frac{3}{2}, \frac{\sqrt{35}}{2})$ and $(\frac{3}{2}, \frac{-\sqrt{35}}{2})$.



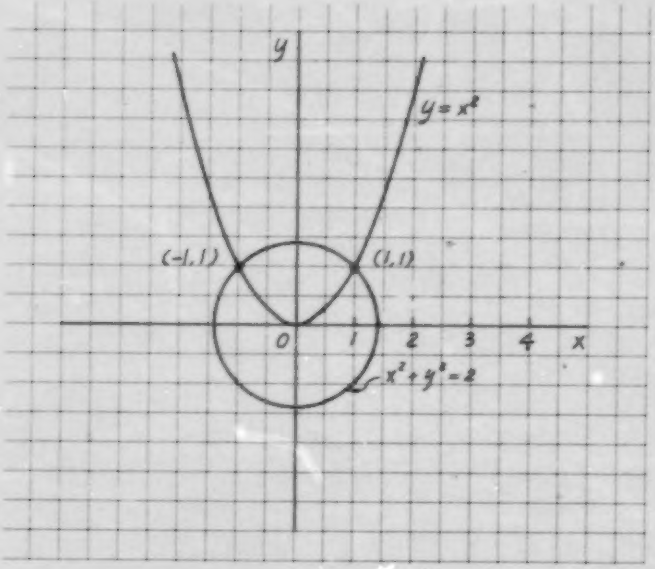
14. The points of intersection are $(3, \sqrt{6})$, $(3, -\sqrt{6})$, $(-3, \sqrt{6})$, and $(-3, -\sqrt{6})$.



15. The points of intersection are (2,2) , (1,1) and (-2,-2).

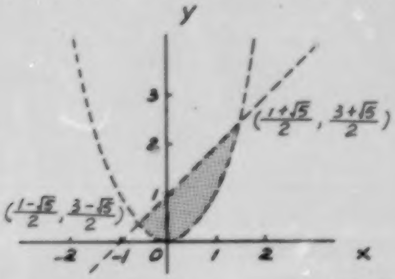

Note that
xy² - xy - 4y + 4
= (y-1)(xy-4)
y
xy - 4 = 0
(2,2)
y - 1 = 0
(1,1)
0
1
2
3
4
x
xy - 4 = 0
(-2,-2)

16. The points of intersection are (1,1) and (-1,1).

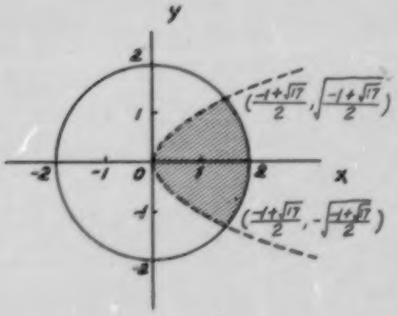

y
y = x²
(-1,1)
(1,1)
0
1
2
3
4
x
x² + y² = 2

For Problems 17 to 19, the intersection is the shaded region.

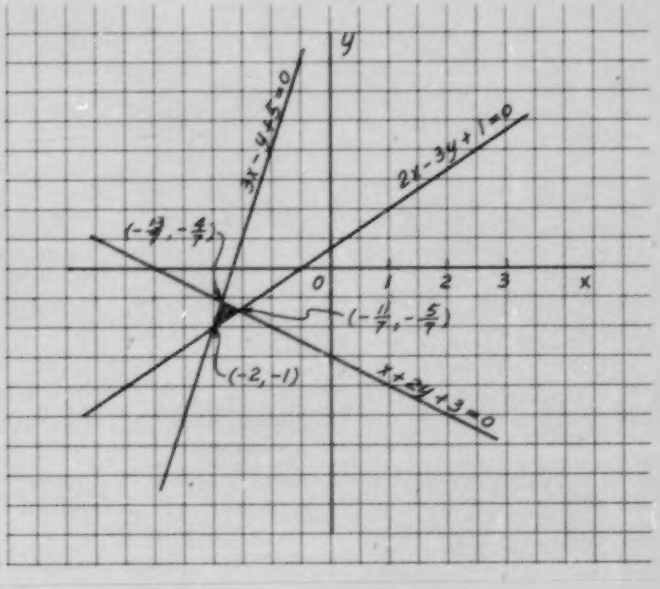
17. $\{(x,y) : y > x^2 \text{ and } y < x + 1\}$



18. $\{(x,y) : x > y^2 \text{ and } x \le \sqrt{y^2 - 4}$



19.



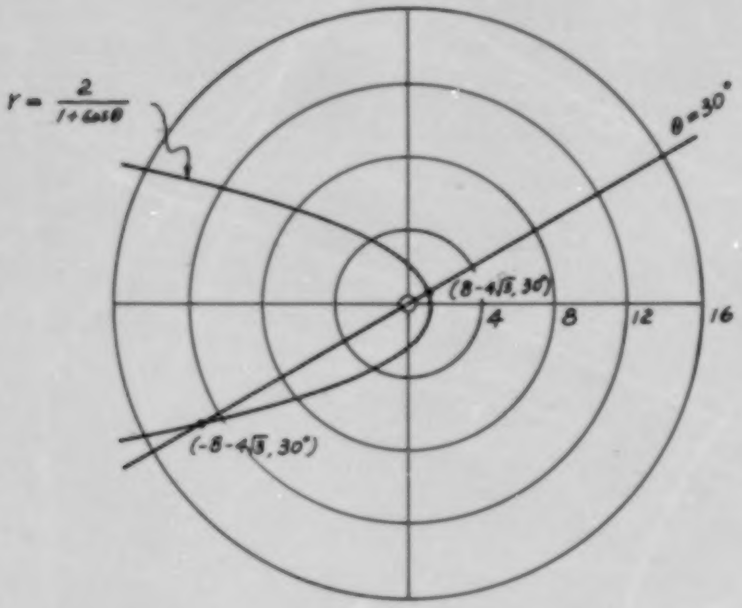
6-6. Intersection of Loci (Polar Coordinates).

Any adequate treatment of this topic must give careful consideration to the special situation at the pole; and to the multiple representations of polar graphs. We have done this in the text, and found the concept of related polar equations particularly useful in finding all intersection points.

Intersections of the graphs of polar inequalities are not treated here because they lead into content beyond the level of this book.

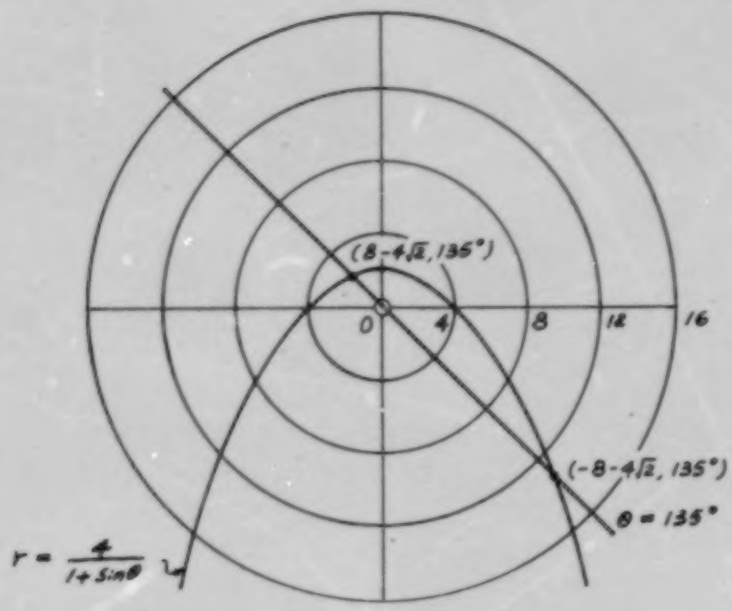
Exercises 6-6

1.



Related equations are $r = -\frac{2}{1 - \cos\theta}$ and $\theta = 210^\circ$.

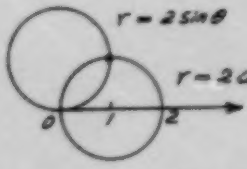
2.


$(8-4\sqrt{2}, 135°)$
0
4
8
12
16
$(-8-4\sqrt{2}, 135°)$
$\theta = 135°$
$r = \frac{4}{1+\sin\theta}$

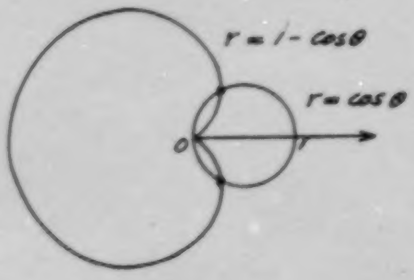
The related equations are $r = -\frac{4}{1 - \sin\theta}$ and $\theta = 315°$.

3. $(0,\theta)$ and $(\frac{\sqrt{2}}{2}, 45°)$

Related equations are the same.


$r = 2\sin\theta$
$r = 2\cos\theta$
0
1
2

4. $(0,\theta)$ $(\frac{1}{2}, 60°)$ $(\frac{1}{2}, 300°)$

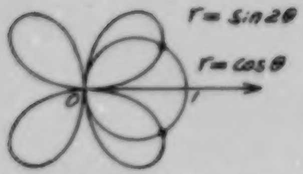

$r = 1 - \cos\theta$
$r = \cos\theta$
0
1

The related equations are

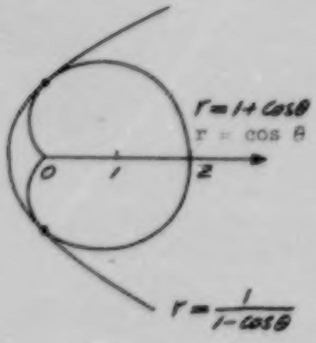
$r = \cos\theta$ and $r = -1 - \cos\theta$.

5. $(0,\theta)$ $(\frac{\sqrt{3}}{2}, 30^\circ)$ $(\frac{\sqrt{3}}{2}, 150^\circ)$

7. $(1,90^\circ)$ $(1,270^\circ)$



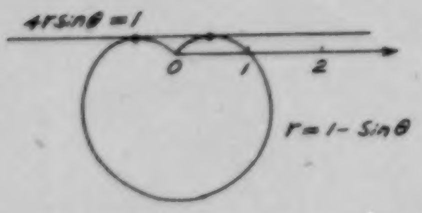
The related equations are $r = \cos\theta$ and
$r = -\sin 2\theta$



$r = \cos\theta - 1$

$r = \frac{-1}{1 + \cos\theta}$

6. $(\frac{1}{2}, 30^\circ)$ $(\frac{1}{2}, 150^\circ)$



The related equations are $-1 -\sin\theta$ and
$4r\sin\theta = 1$.

6-7. Families of Curves.

This topic is a necessary foundation for parts of the usual calculus courses. We expand the idea of linear combinations of functions and treat families of curves more generally. A one-parameter family is related to the physical concept of one "degree of freedom" and so on. It is instructive to develop this briefly, showing how the restriction of each degree of freedom is equivalent to the assignment of a specific value to one parameter. Thus, drawing a circle on the blackboard involves three degrees of freedom: locate it horizontally, locate it vertically, determine its radius. These determinations are made, and the degrees of freedom are restricted, by assigning specific values respectively to a, b, r, in the three-parameter family: $(x - a)^2 + (y - b)^2 = r^2$.

The method of choosing a particular member of a family is often more complicated than assigning numerical values to the parameters. The particular member may be determined by a condition whose application may be quite indirect. There will be future applications of the method shown for picking out a tangent line, by imposing the condition that such a line has a double contact point. Algebraically this means that the equation which gives the abscissas of the intersection points must have multiple roots. If the equation is quadratic, as in the text, this condition means that the discriminant of the quadratic equation must equal zero. This is what gives us the equation from which we pick out the values of the parameter for the members of the family that satisfy our condition.

The review exercises at the end of the chapter furnish many opportunities to write families of curves, mostly lines and circles. We did not include exercises in which the student is asked to pick out a particular member of the family to satisfy a given condition. These are simple to improvise, and may take any of the following forms: Find the member of the family which goes through a given point; find the member of the family that has a given slope; find the member of the family that is tangent to a given circle; etc. It is instructive to consider each family as a set of curves; then the question of finding a particular curve that satisfies two conditions is equivalent to the question of finding the intersection of two sets of curves. Note carefully that we use the word "intersection" here to mean the curve (or curves) common to the two sets of curves that comprise the two families.

Exercises 6-7

1. $x = a$.
2. $y = a$.
3. $y + 1 = k(x - 2)$.
4. $y = kx + b$
5. $(x + 1)^2 + (y - 2)^2 = a^2$
6. $(x - a)^2 + (y - b)^2 = 16$
7. $\frac{x}{a} = y^2$.
8. $3x - 4y + k = 0$.
9. $x - 2y + k = 0$.
10. $x \cos t + y \sin t - 5 = 0$.
11. $x \cos t + y \sin t - p = 0$, $p > 5$.
12. $x^2 + y^2 - 2hx - 2ky = 0$, $h^2 + k^2 = 36$; or

    $x^2 + y^2 - 2hx - 2\sqrt{36 - h^2}\, y = 0$.

13. $(x - h)^2 + (y - k)^2 = 1$, $h^2 + k^2 > 1$.

14. $y = 4x - 12$.

15. $3x - 3y + 20 = 0$.

16. $4x + 5 = 0$.

17. The family of all lines through the intersection of the given lines is represented $a(x - 2y + 3) + b(x + 3y - 2) = 0$. Picking a and b so that $(1,1)$ lies on the line and simplifying we get $y = 1$.

18. $(a + b)x^2 + (a + b)y^2 - 2(a - b)x + 4by - 35a - 44b = 0$
For each pair (a,b) this equation represents a circle through the intersection of the given two. The whole family of circles is called a coaxial family. Their centers are all on the perpendicular bisector of the common chord of any two of these circles. If $a = 1$, $b = 0$ we get the first circle; if $a = 0$, $b = 1$, we get the second circle; if $a = -b$ we get a line, the line along this common chord. This line may be considered a degenerate circle. It has the property that from any point on it the tangents to all members of the coaxial family have equal lengths.

19. $x + 3y - 7 = 0$.

20. $3x + 4y - 15 = 0$.

21. $y - 5 = \frac{-72}{a^2}(x - 2)$ a represents the x-intercept: $\frac{72}{a}$, the y-intercept. Thus $\frac{-72}{a^2}$ is the slope, and $\frac{1}{2}(a)(\frac{72}{a})$ the area of the triangle in the first quadrant.

22. $x = 5$ and $3x - 4y + 25 = 0$. Any line through the intersection of the given lines can be represented by $a(y - 10) + b(2x - y) = 0$, that is, by $2bx + (a - b)y - 10a = 0$. The distance from such a line to the origin is $\frac{|2b(0) + (a - b)0 - 10a|}{\sqrt{4b^2 + (a - b)^2}}$ that is $\frac{|10a|}{\sqrt{a^2 - 2ab + 5b^2}}$.

If this distance is 5 then $|2a| = \sqrt{a^2 - 2ab + 5b^2}$. Thus $3a^2 + 2ab - 5b^2 = 0$. Set $a = tb$ and we get $3t^2 + 2t - 5 = 0$ and $t = 1$ or $-\frac{5}{3}$. These give us the solutions above.

Review Exercises

1. First, we find the coordinates of the intersections of the two given lines with all lines parallel to the x-axis. For a line parallel to the x-axis, the y-coordinates of the two intersections are the same. Thus we have $x_1 - 8 = -y_1$ and $2x_2 - 1 = y_2 = y_1$, or $x_2 - \frac{1}{2} = \frac{y_1}{2}$. Adding gives $\frac{x_1 + x_2}{2} = \frac{17 - y_1}{4}$. Hence the equation of the desired locus is $x = \frac{17 - y}{4}$ or $y = -4x + 17$.

2. In the same manner as in 1 above we find the equation

$$y = \frac{1}{2}x + \frac{7}{2}$$

3. (a) $d(P,A) = \sqrt{(x + 4)^2 + y^2} = 2d(P,B) = 2\sqrt{(x - 4)^2 + y^2}$ which gives us upon simplifictaion $3x^2 + 3y^2 - 40x + 48 = 0$ which is the equation of a circle.

   (b) $\sqrt{(x + 4)^2 + y^2} + \sqrt{(x - 4)^2 + y^2} = 10$ which gives $\frac{x^2}{25} + \frac{y^2}{9} = 1$ which is the equation of an ellipse.

   (c) $\sqrt{(x + 4)^2 + y^2} - \sqrt{(x - 4)^2 + y^2} = 2$ which gives $x^2 - \frac{y^2}{15} = 1$ which is the equation of a hyperbola.

   (d) If the lines are perpendicular, the product of their slopes is -1, therefore

   $\frac{y - 0}{x + 4} \cdot \frac{y - 0}{x - 4} = -1$, $\therefore$ $y^2 = 16 - x^2$, or $x^2 + y^2 = 16$.

   This is an equation of a circle.

   (e) $\frac{y - 0}{x + 4} = 2 \cdot \frac{y - 0}{x - 4}$, $\therefore$ $y = 0$; or $x = -12$.

   The locus is the pair of lines whose equations are given above.

   (f) $\frac{y - 0}{x + 4} = 1 + \frac{y - 0}{x - 4}$, therefore $xy - 4y = x^2 - 16 + xy + 4y$.

   This equation may be written $x^2 + 8y - 16 = 0$, and is an equation of a parabola.

(g) Let α be the inclination of $\overleftrightarrow{PA}$ and β be the inclination of $\overleftrightarrow{PB}$. If m_1, m_2 are their respective slopes then

$\tan \angle APB = \tan(\beta - \alpha)$

$$\therefore 1 = \frac{m_2 - m_1}{1 + m_2 m_1} \quad \text{or} \quad 1 = \frac{\frac{y}{x+4} - \frac{y}{x-4}}{1 + \frac{y}{x+4} \cdot \frac{y}{x-4}} = \frac{8y}{x^2 - 16 + y^2} .$$

This equation may be written more simply as $x^2 + (y - 4)^2 = 32$, and is an equation of a circle.

(h) In the same way as in (g) above, we have

$\tan \angle APB = \tan 60^\circ = \sqrt{3} = \dfrac{m_2 - m_1}{1 + m_2 m_1} = \dfrac{8y}{x^2 - 16 + y^2}$. This equation may be written $x^2 + y^2 - \dfrac{8}{\sqrt{3}} y = 16$ and is an equation of a circle.

(i) Area $= \frac{1}{2}hb$; $d(A,B) = 8$ $\therefore$ $\frac{1}{2}8h = 20$ $\therefore$ $h = 5$.

Since the distance from P to the x-axis must be 5, the locus of P is the pair of lines whose equations are $y = +5$, $y = -5$.

(j) $\sqrt{(x + 4)^2 + y^2} < \sqrt{(x - 4)^2 + y^2}$ means that $x < 0$.

4. Let $M = (r,s)$ be the midpoint of $\overline{AP}$. Then $r = \dfrac{x + 6}{2}$, $s = \dfrac{y + 0}{2}$;

$\therefore$ $x = 2r - 6$ and $y = 2s$. Since $P = (x,y)$ is on the circle $x^2 + y^2 = 36$, we have $(2r - 6)^2 + (2s)^2 = 36$ or $r^2 - 6r + 9 + s^2 = 9$. This may be written $r^2 + s^2 - 6r = 0$ or $x^2 + y^2 - 6x = 0$ and is an equation of a circle, which is the required locus.

5. Let $P = (r,s)$. Then $x = \dfrac{r + 0}{2}$, $y = \dfrac{s + 5}{2}$. Since $x^2 + y^2 = 25$,

$\therefore$ $(\frac{r}{2})^2 + (\frac{s + 5}{2})^2 = 25$, or $r^2 + (s + 5)^2 = 100$. This equation may be written $x^2 + y^2 + 10y - 75 = 0$ and is an equation of a circle, which is the required locus.

6. Let D = (x,y) , E = (20,t) and M , the midpoint of $\overline{DE}$ = (r,s) . From similar triangles, $\frac{y}{x + 10} = \frac{t}{30}$, ∴ $t = \frac{30y}{x + 10}$. We have $r = \frac{x + 20}{2}$, $s = \frac{y + t}{2}$; ∴ $x = 2r - 20$, $y = 2s - t = 2s - \frac{30y}{x + 10}$. These equations yield $x = 2(r - 10)$, $y = \frac{2s(r - 5)}{r + 10}$. Since D is on the circle $x^2 + y^2 = 100$, we have $(2(r - 10))^2 + (\frac{2s(r - 5)}{r + 10})^2 = 100$, or,

$(r - 10)^2 + \frac{s^2(r - 5)^2}{(r + 10)^2} = 25$. This may be written

$(r - 10)^2(r^2 - 20r + 100 - 25) + s^2(r - 5)^2 = 0$, or

$(r + 10)^2(r - 5)(r - 15) + s^2(r - 5)^2 = 0$; which is equivalent to

$(r - 15)(r + 10)^2 + (r - 5)s^2 = 0$. Therefore an equation for the required locus is $(x - 15)(x + 10)^2 + (x - 5)y^2 = 0$.

However, a much simpler solution is available in polar coordinates. Take the pole at C and the polar axis to the right along the x-axis. Let D = (p,θ) , E = (q,θ) , and M , the midpoint of $\overline{DE}$ = (r,θ) . Then $\frac{p}{20} = \cos\theta$, $\frac{30}{q} = \cos\theta$ and $r = \frac{1}{2}(p + q) = \frac{1}{2}(20\cos\theta + 30\sec\theta)$. Therefore an equation for the required locus is $r = 10\cos\theta + 15\sec\theta$. We may show the equivalence of these two solutions by using the relationship: $r^2 = x'^2 + y'^2$, $\cos\theta = \frac{x'}{\sqrt{x'^2 + y'^2}}$; and $x' = x + 5$, $y^1 = 5$.

The computation is elementary but tedious.

Any line parallel to $y = 3x + 5$ has an equation $y = 3x + d$, and will intersect the circle $x^2 + y^2 - 4x + 8y = 0$ in two points whose abscissas are the roots of $x^2 + (3x + d)^2 - 4x + 8(3x + d) = 0$. If the midpoint of this chord has coordinates (r,s) , then $r = \frac{1}{2}$ the sum of the abscissas of the endpoint, that is, $\frac{1}{2}$ the sum of the roots of this equation, and this result can be found from the coefficients directly without solving the equation. Thus $10x^2 + (20 + 6d)x + d^2 + 8d = 0$, and $r = -\frac{10 + 3d}{10}$, ∴ $s = 3r + d = -\frac{30 - d}{10}$. Eliminating d from these two equations yields $r + 3s + 10 = 0$, therefore an equation for the required locus is $x + 3y + 10 = 0$.

8. In the same manner as in Exercise 7, we find $x - 9y = 0$ as an equation for the locus.

9. (a) The line $\frac{x}{a} + \frac{y}{b} = 1$ has intercepts a and b. The conditions of the problem requires that $ab = \pm 24$. Therefore $\frac{x}{a} \pm \frac{ay}{24} = 1$ is a pair of equations representing two one-parameter families of lines, the solution we require. Of course $a \neq 0$.

(b) As in 9(a), we need $\frac{x}{a} + \frac{y}{b} = 1$ and $a + b = 6$, that is $\frac{x}{a} + \frac{y}{6 - a} = 1$ with $a \neq 0, 6$. We may consider that a line parallel to an axis has just one intercept whose "sum" is itself; in which case we may include in our solution the lines $x = 6$; and $y = 6$.

(c) $(x - a)^2 + (y - b)^2 = a^2$

(d) $(x - a)^2 + (y - b)^2 = b^2$

(e) The distance from the center (a,b) of one such circle to the line $4x + 3y - 2 = 0$ is $\frac{4a + 3b - 2}{5}$, and, by the conditions of the problem, equals ± 1. The centers must lie therefore on the lines $4x + 3y - 7 = 0$ and $4x + 3y + 3 = 0$, which are parallel to the original lines. The families of circles are therefore $(x - a)^2 + (y - b)^2 = 1$ where (a,b) must satisfy one of the equations of the lines just found. In terms of a single parameter the answers are $(x - a)^2 + (y - \frac{7}{3} + \frac{4}{3}a)^2 = 1$, and $(x - a)^2 + (y + 1 + \frac{4}{3}a)^2 = 1$.

(f) The two families are $(x - a)^2 + (y - \frac{5r + 2}{3} + \frac{4a}{3})^2 = r^2$,

$(x - b)^2 + (y + \frac{5r - 2}{3} + \frac{4}{3}b)^2 = r^2$.

(g) $(x - a)^2 + (y - b)^2 = 36$ where $a^2 + b^2 < 36$.

(h) $(x - a)^2 + (y - b)^2 = a^2 + b^2$.

(i) The distance from $P = (x,y)$ on the circle to the center (a,b) must equal the distance from $(12,5)$ to the same center. Therefore the circles we want have equation:

$$(x - a)^2 + (y - b)^2 = (12 - a)^2 + (5 - b)^2 .$$

(j) $(x - a)^2 + (y - b)^2 = r^2$ where $a^2 + b^2 < r^2$.

(k) $(x - a)^2 + (y - b)^2 = 25$ where $a^2 + b^2 > 25$.

(ℓ) The two families are $(x - g)^2 + \left(y - \frac{d\sqrt{a^2 + b^2} - c - ag}{b}\right)^2 = d^2$

and $(x - h)^2 + \left(y + \frac{d\sqrt{a^2 + b^2} + c + ah}{b}\right)^2 = d^2$ where g and h are arbitrary.

(m) A point (a,b) on a bisector of the angles formed by the two lines must be equidistant from them, therefore

$\left|\frac{3a - 4b + 5}{5}\right| = \left|\frac{4a - 3b + 9}{5}\right|$. These bisectors have, therefore, the equations $3x - 4y + 5 = 4x - 3y + 9$; and $3x - 4y + 5 = -4x + 3y - 9$; that is, $x \quad y + 4 = 0$, and $x - y + 2 = 0$.

(Note that these lines are perpendicular to each other.)

Therefore $b = -a - 4$, or $b = a + 2$. The families of circles:

$(x - a)^2 + (y - b)^2 = r^2$, become

$(x - a)^2 + (y + a + 4)^2 = (\frac{7}{5}a + \frac{21}{5})^2$; and

$(x - a)^2 + (y - a - 2)^2 = (\frac{1}{5}a + \frac{3}{5})^2$.

(n) The families are $(x - g)^2 + (y - h)^2 = \frac{(a_1g + b_1h + c_1)^2}{a_1^{\,2} + b_1^{\,2}}$ where

$$h = \frac{\dfrac{a_2g + c_2}{\sqrt{a_2^{\,2} + b_2^{\,2}}} - \dfrac{a_1g + c_1}{\sqrt{a_1^{\,2} + b_1^{\,2}}}}{\dfrac{b_1}{\sqrt{a_1^{\,2} + b_1^{\,2}}} - \dfrac{b_2}{\sqrt{a_2^{\,2} + b_2^{\,2}}}}$$

and for the other family

$$h = -\frac{\dfrac{a_2g + c_2}{\sqrt{a_2^{\,2} + b_2^{\,2}}} + \dfrac{a_1g + c_1}{\sqrt{a_1^{\,2} + b_1^{\,2}}}}{\dfrac{b_1}{\sqrt{a_1^{\,2} + b_1^{\,2}}} + \dfrac{b_2}{\sqrt{a_2^{\,2} + b_2^{\,2}}}} .$$

(o) $(x - a)^2 + (y - b)^2 = r^2$ where $b^2 \leq r^2$.

(p) $(x - a)^2 + (y - b)^2 = r^2$ where $a^2 > r^2$.

(q) $(x - g)^2 + (y - h)^2 = r^2$ where $\frac{|ga + hb + c|}{\sqrt{a^2 + b^2}} > r$.

(r) $(x - a)^2 + (y - b)^2 = r^2$ where $0 < r$ and $\sqrt{a^2 + b^2} + r < 10$.

(s) $(x - a)^2 + (y - b)^2 = r^2$ where $r > 0$ and $|r - 1| \leq \sqrt{a^2 + b^2} \leq r + 1$.

(t) $ax + by + d = 0$ where $\frac{d}{\sqrt{a^2 + b^2}} \leq 1$.

(u) $(x - a)^2 + (y - b)^2 = r^2$ where $0 < a$, $0 < b$, $a + b < 10$, and

$r <$ the smallest of $\left(a , b , \frac{|a + b - 10|}{\sqrt{2}}\right)$.

(v) $(x - a)^2 + (y - b)^2 = r^2$ where $r^2 > a^2 + (10 - b)^2$,

$r^2 > a^2 + b^2$, and $r^2 > (a - 10)^2 + b^2$.

(w) $(x - a)^2 + (y - b)^2 = r^2$ where $\sqrt{a^2 + b^2} + r = 10$.

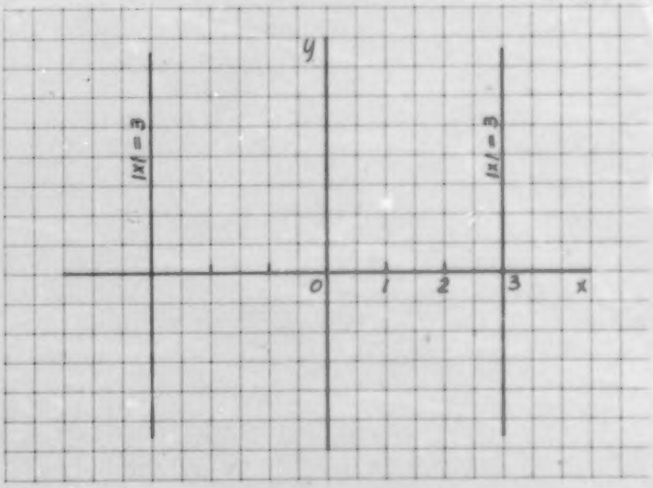
(x) $(x - a)^2 + (y - b)^2 = r^2$ where $\sqrt{a^2 + b^2} - r = 10$.

(y) $(x - a)^2 + (y - b)^2 = r^2$ where $r - \sqrt{a^2 + b^2} = 10$.

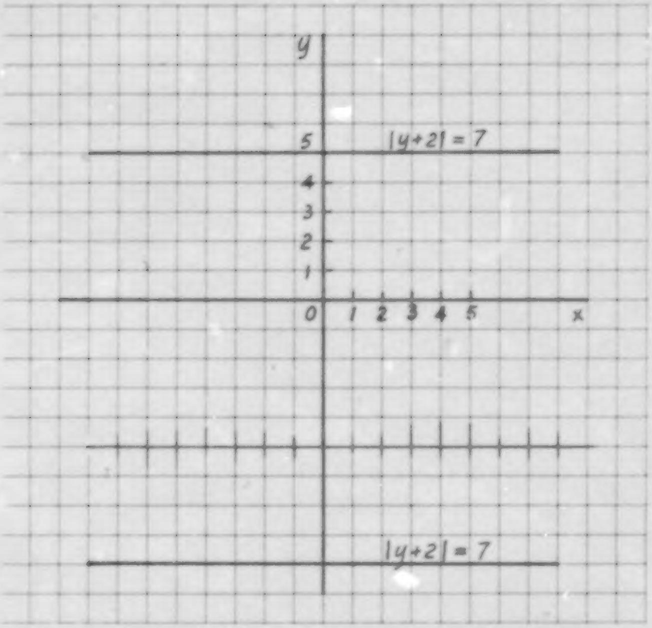
(z) $(x - g)^2 + (y - h)^2 = R^2$ where $\frac{|ag + bh + c|}{\sqrt{a^2 + b^2}} = R$ and

$(r - g)^2 + (s - h)^2 = R^2$.

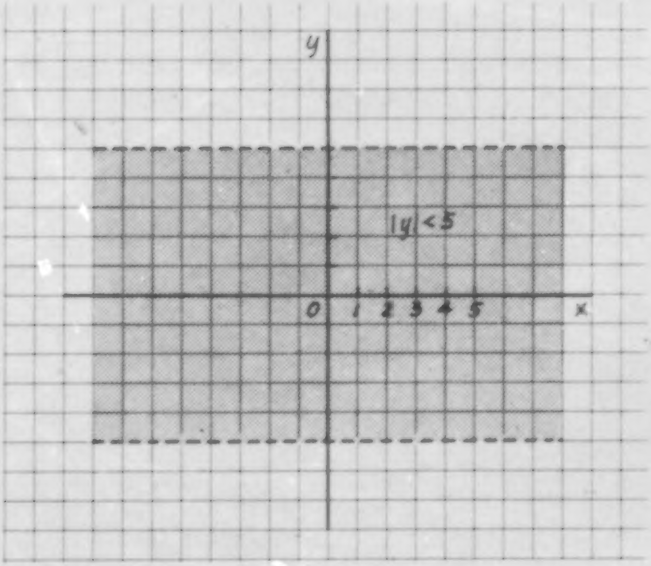
10. (a)



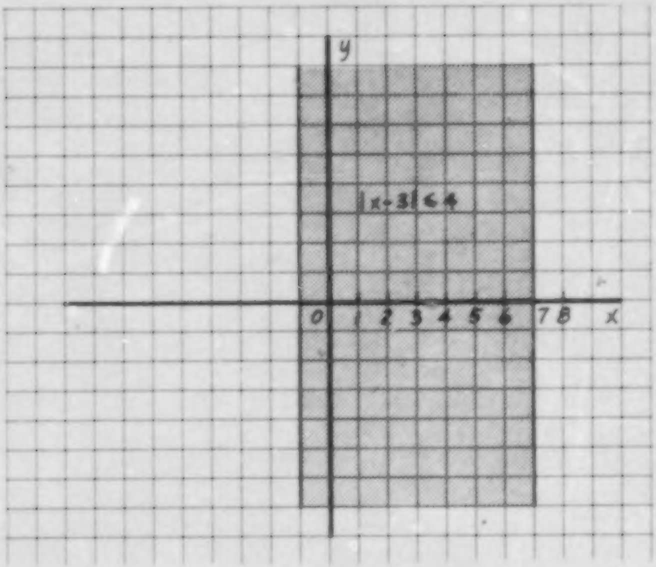
(b)



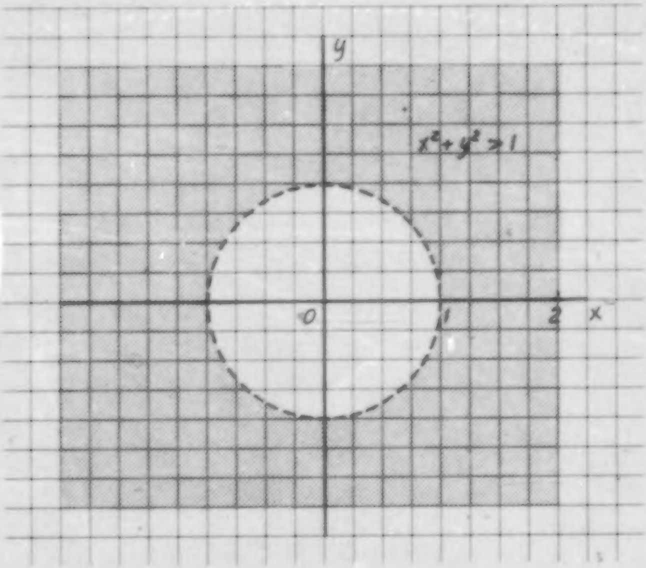
(c)



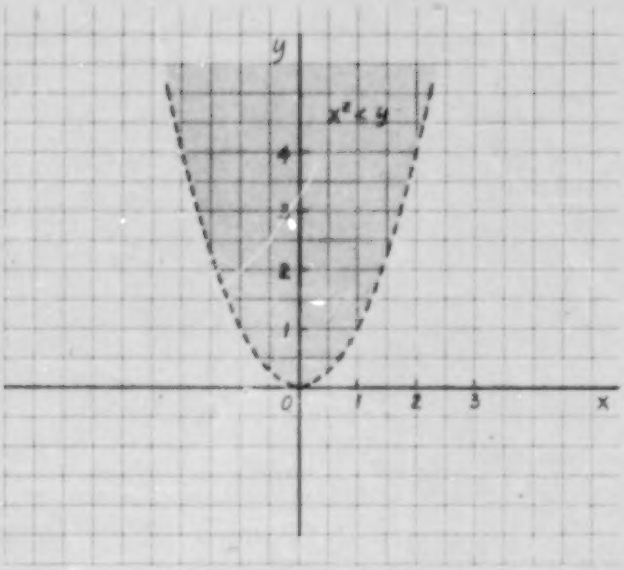
(d)



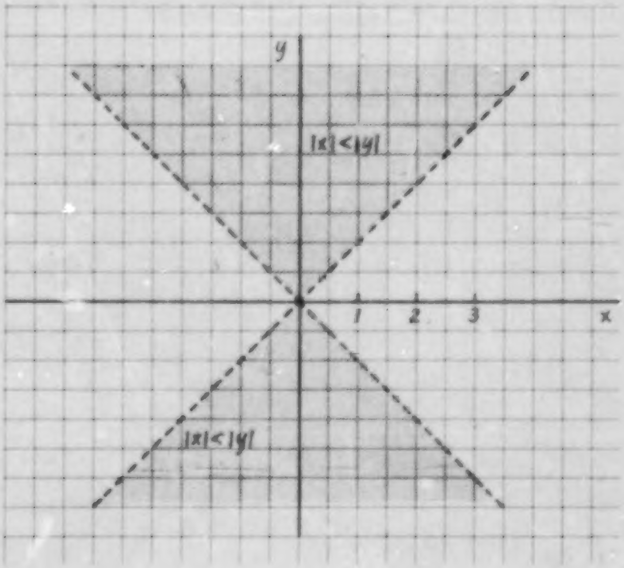
(e)



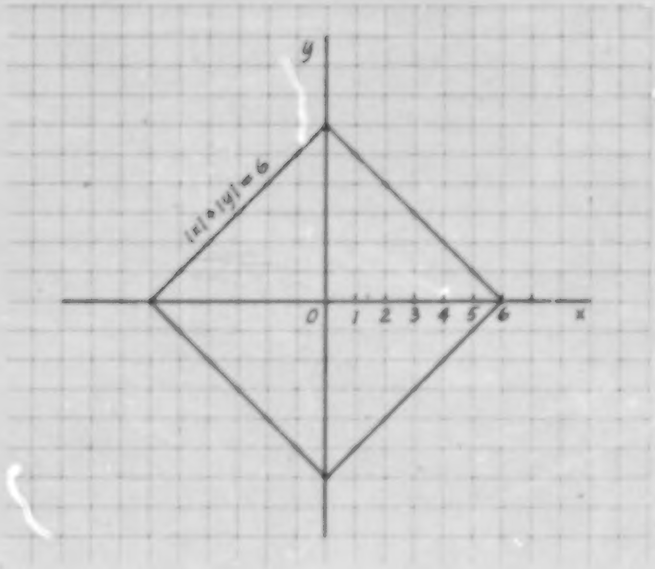
(f)



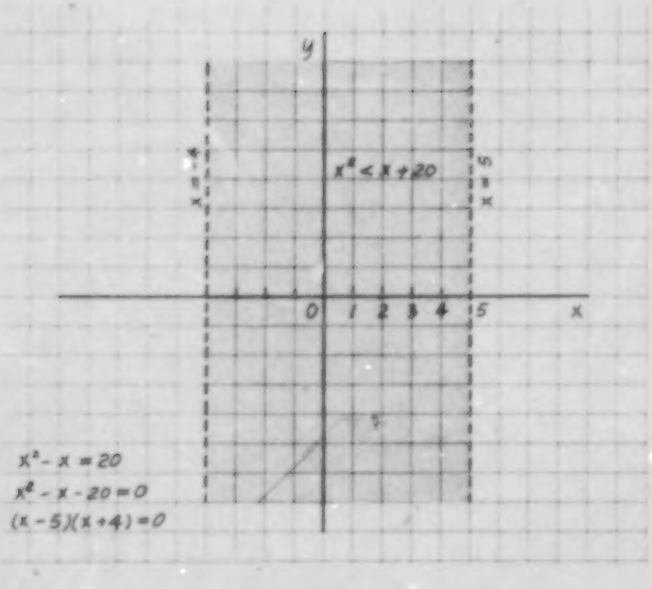
(g)



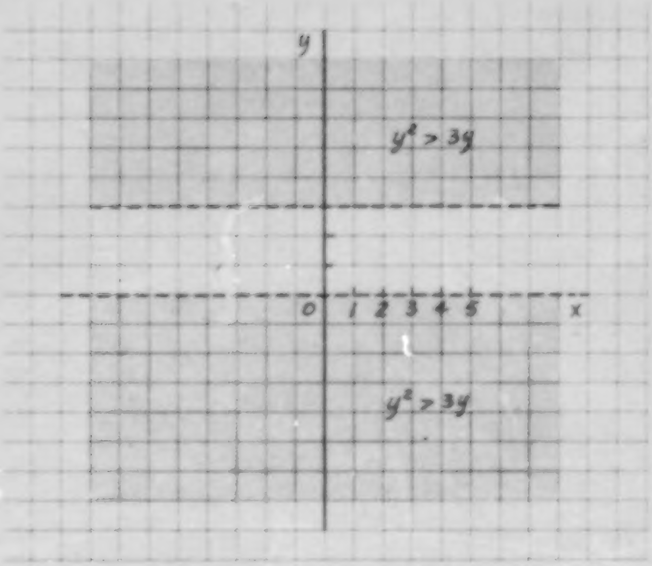
(h)



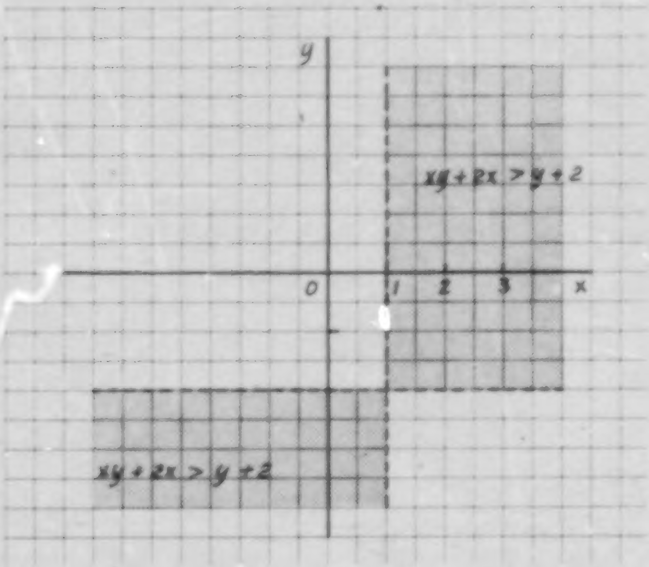
(i)



(j)

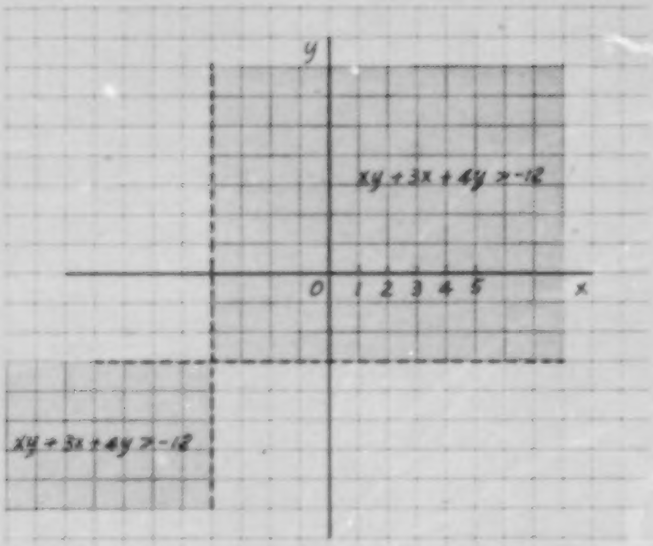


(k)

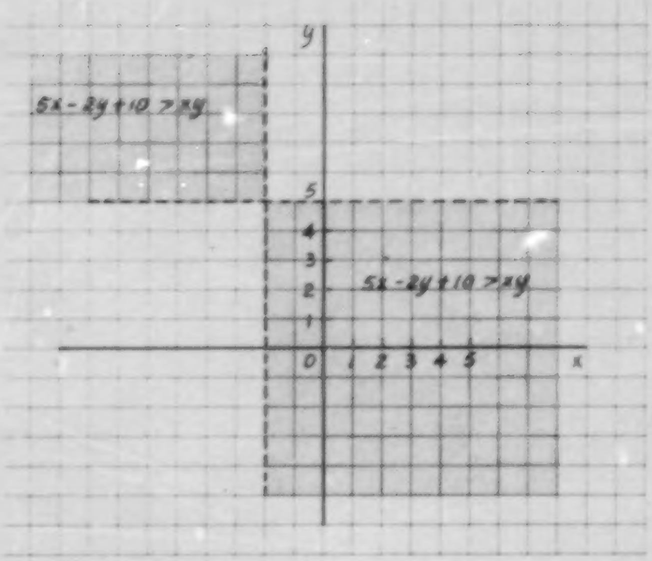


The given condition is equivalent to $xy + 2x - y - 2 > 0$ or $(x - 1)(y + 2) > 0$. Therefore both factors must be positive, or both factors must be negative. These conditions require $x > 1$ and $y > -2$; or $x < 1$ and $y < -2$.

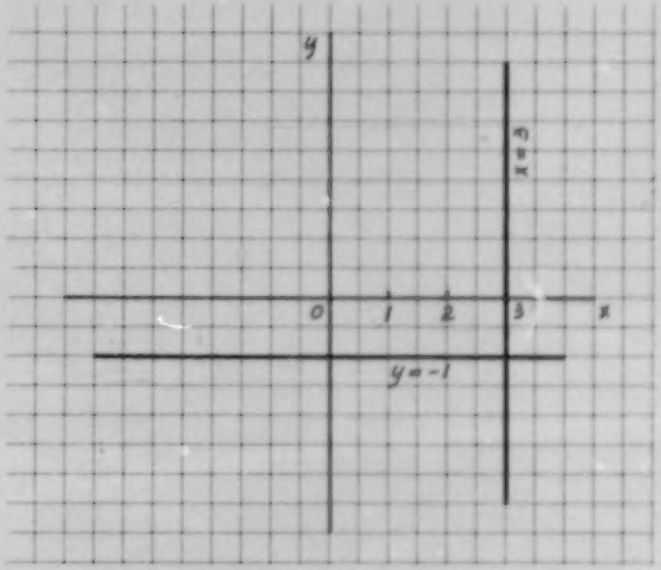
(l)


y
xy + 3x + 4y > -12
0
1
2
3
4
5
x
xy + 3x + 4y > -12

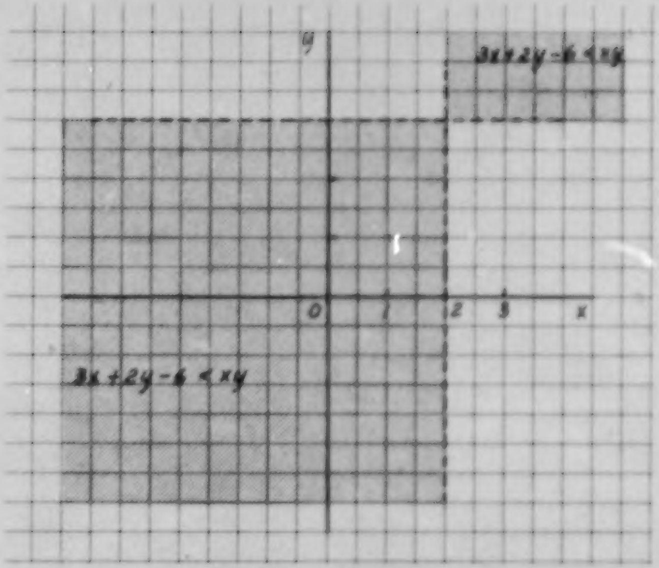
(m)


y
5x - 2y + 10 > xy
5
4
3
2
1
5x - 2y + 10 > xy
0
1
2
3
4
5
x

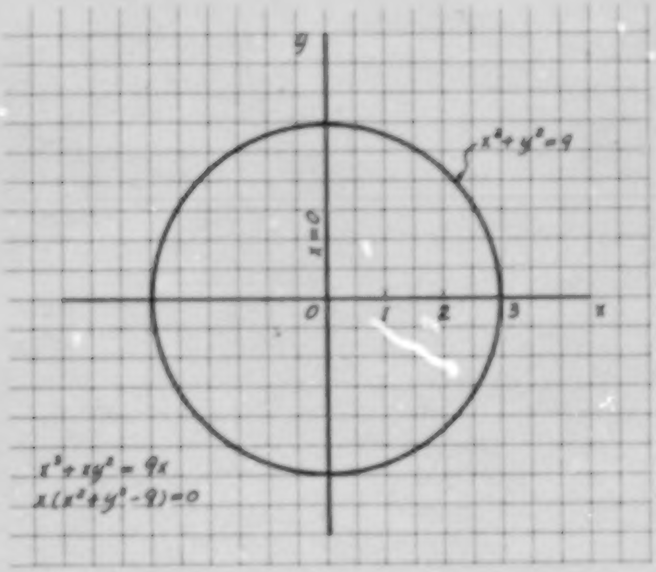
(n)



(o)

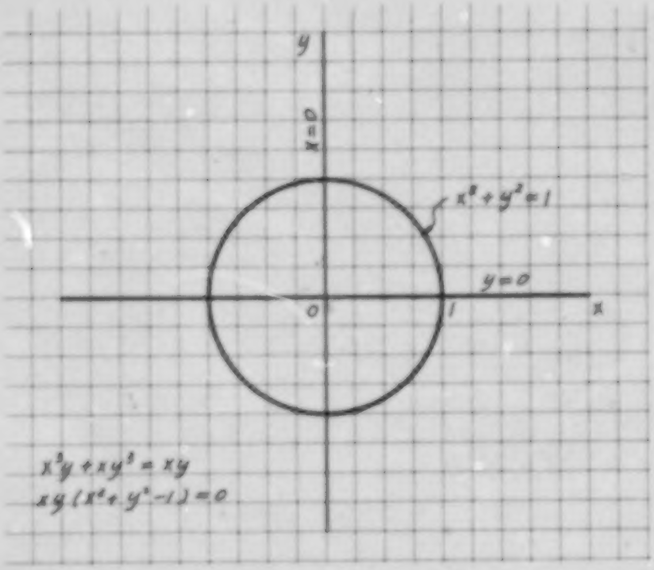


(p)



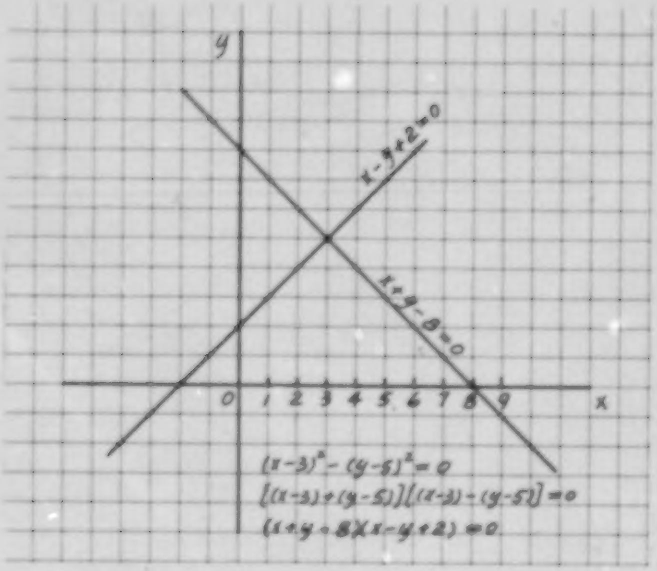
The graph is the set of all points on the circle, and all points on the y-axis, as shown.

(q)

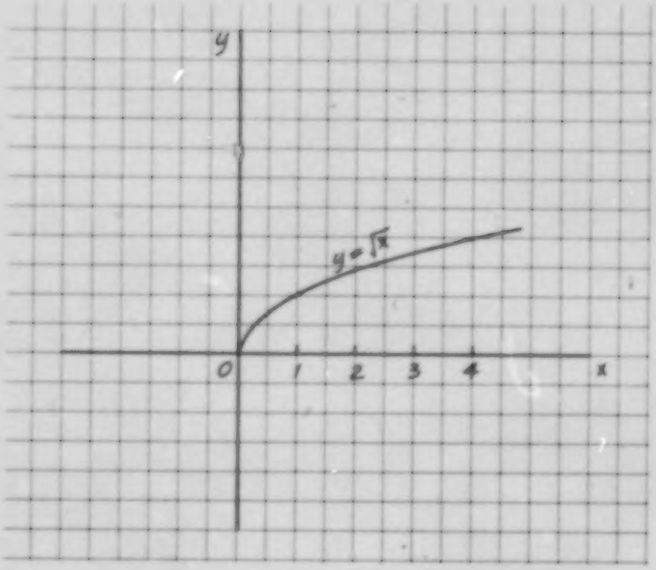


The graph is the set of all points on the circle, all points on the x-axis, and all points on the y-axis, as shown.

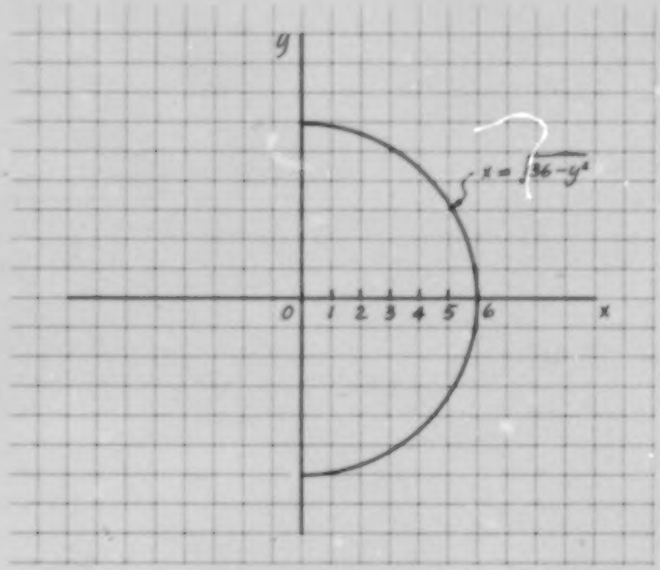
(r)


y
x - y + 2 = 0
x + y - 8 = 0
0
1
2
3
4
5
6
7
8
9
x
(x-3)^2 - (y-5)^2 = 0
[(x-3)+(y-5)][(x-3)-(y-5)] = 0
(x+y-8)(x-y+2) = 0

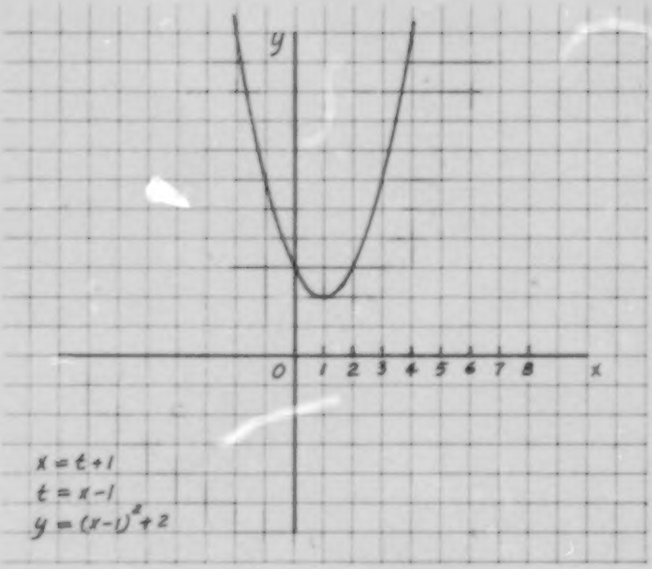
(s)


y
y = √x
0
1
2
3
4
x

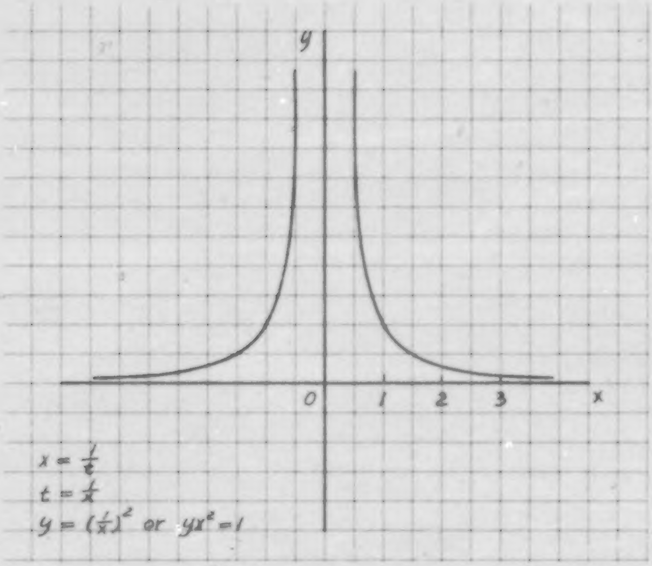
(t)



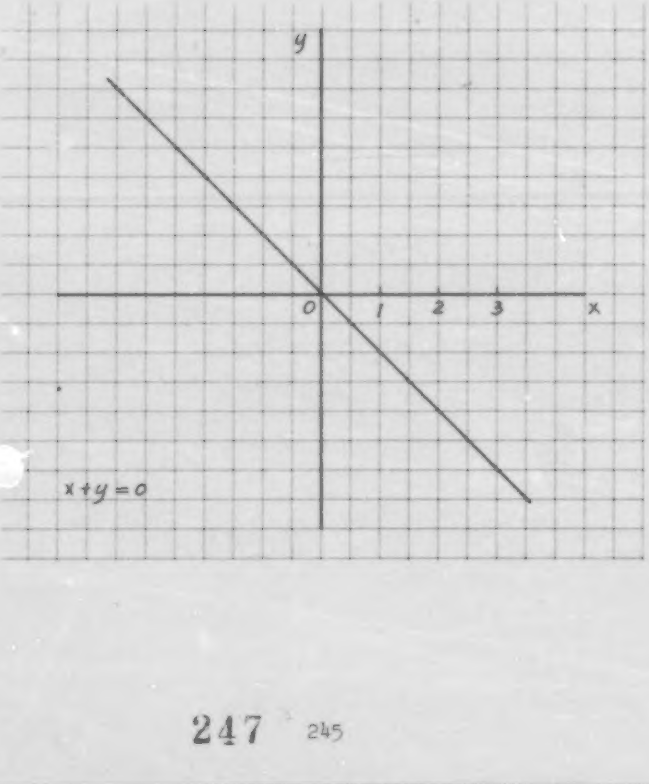
11. (a)



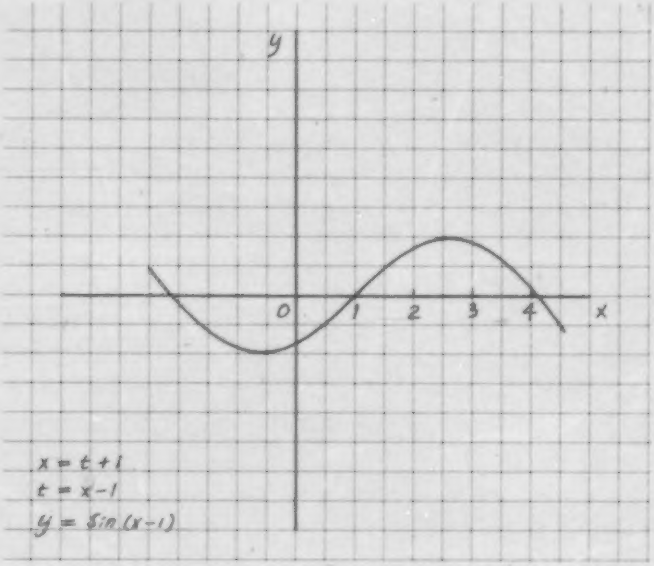
(b)



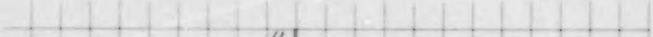
(c)



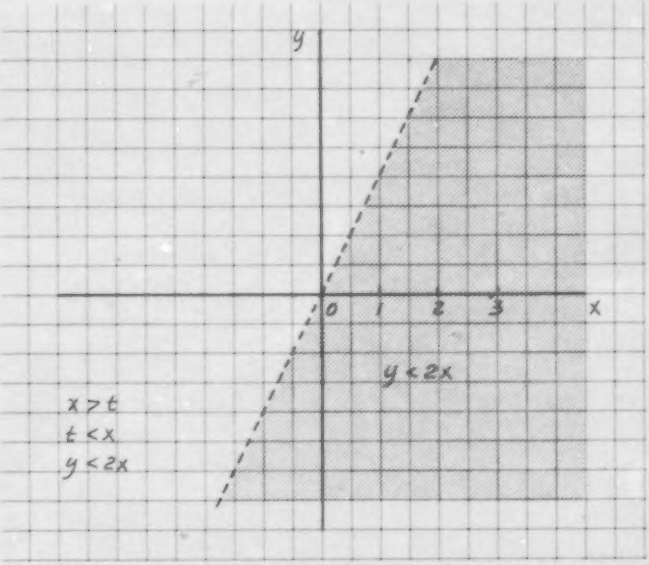
(d)



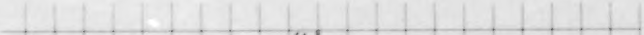
(e)



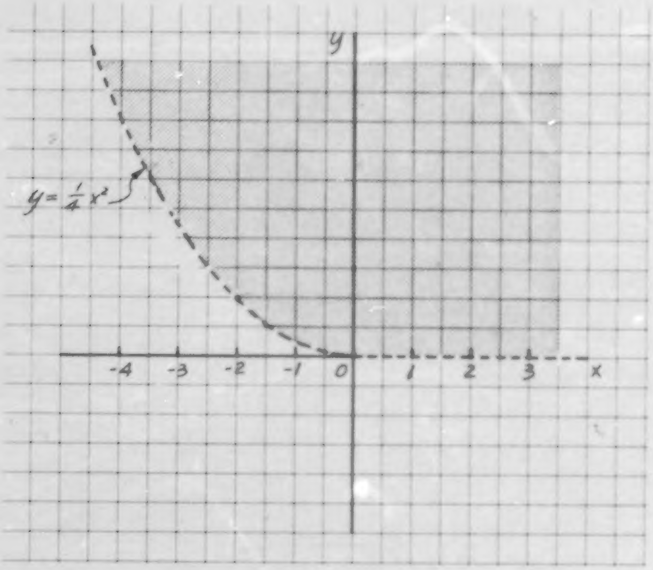
(f)



(g)

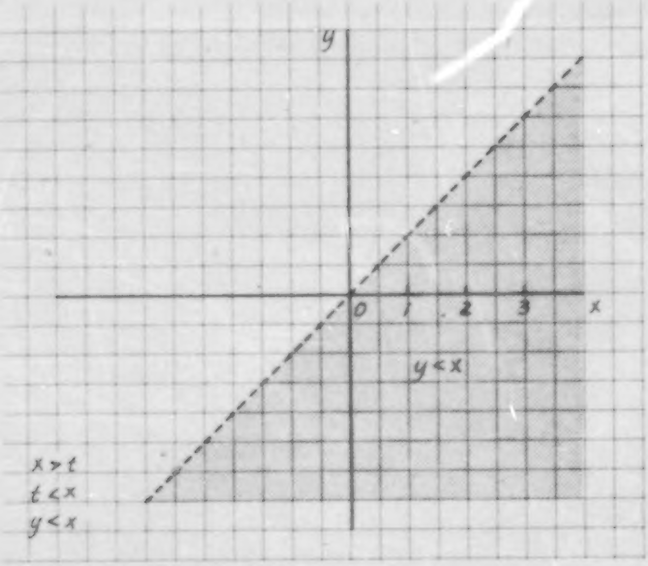


(h)

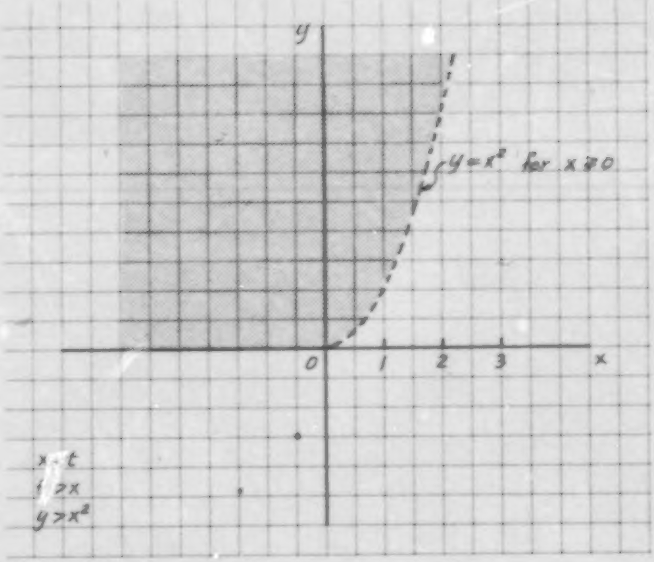


From $y = t^2$ we have $t = \pm\ y$ and $x > \pm 2\ y$. There $y \geq 0$, and our locus is the set of all points *above* the x-axis and to the *right* of either branch of the parabola $x^2 = 4\ y$. It is sufficient to take all points which are both above the x-axis, and to the right of the left branch, as shown.

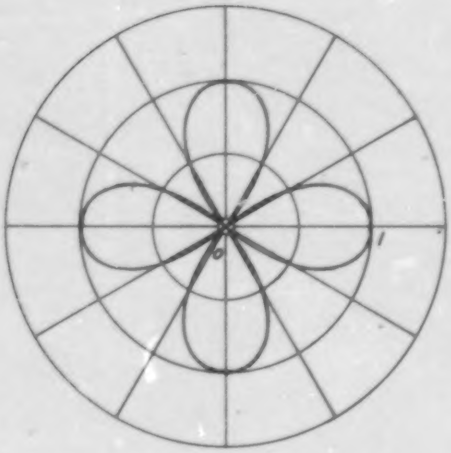
(i)



(j)

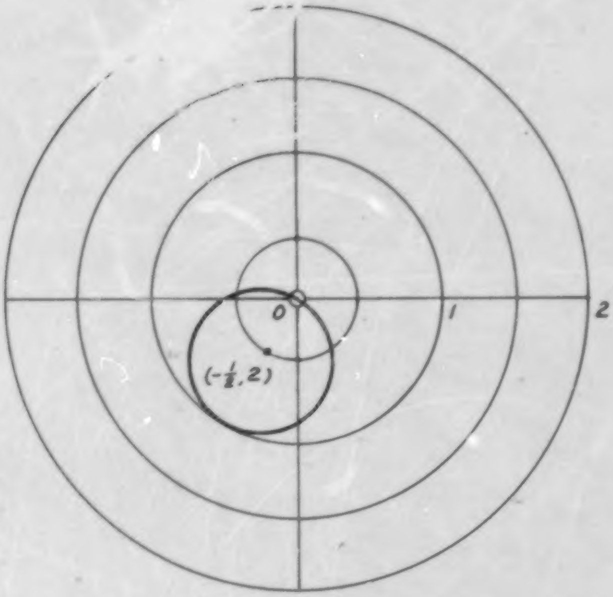


12. (a)



$r = \cos 2\theta$. Four leafed rose, symmetric with respect to the pole, and with respect to the lines $\theta = 0^\circ$, $\theta = 90^\circ$, $\theta = 45^\circ$, $\theta = 135^\circ$. Symmetry with respect to the last two lines can be shown if we use the related polar equation, thus: the points $(r, 45^\circ - \alpha)$, $(r, 45^\circ + \alpha)$ are symmetrically situated with respect to the line $\theta = 45^\circ$, but $\cos 2(45^\circ - \alpha) \neq \cos 2(45^\circ + \alpha)$. However, the point $(r, 45^\circ + \alpha)$ is on the curve for which we have the equation $r = -\cos 2(\theta - 180^\circ)$, and we now have $\cos 2(45^\circ - \alpha) = -\cos 2(45^\circ + \alpha + 180^\circ)$, as can easily be shown. In the same way we could show symmetry with respect to the line $\theta = 135^\circ$ by showing $\cos 2(135^\circ - \alpha) = -\cos 2(135^\circ + \alpha + 180^\circ)$.

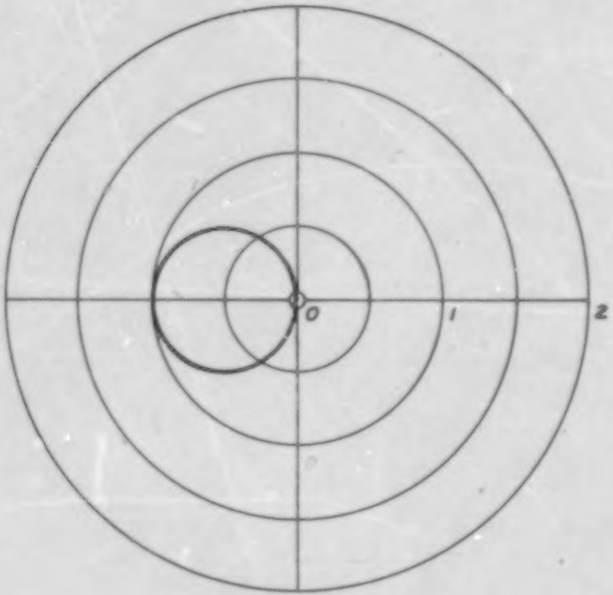
(b)


0
1
2
$(-\frac{1}{2}, 2)$

$r = \cos(\theta + 2)$

Circle of radius $\frac{1}{2}$, with center at $(\frac{1}{2}, -2)$.

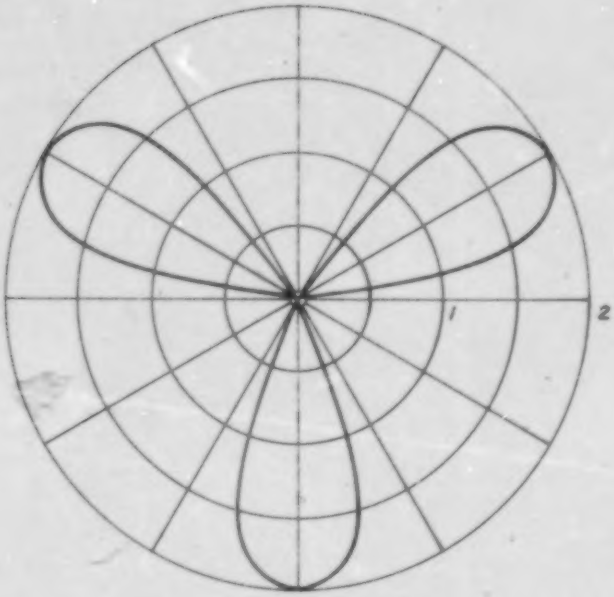
(c)



$r = \sin\left(\theta - \frac{\pi}{2}\right) = -\cos\theta$

Circle of radius $\frac{1}{2}$ with center at $\left(\frac{1}{2}, \pi\right)$.

(d)

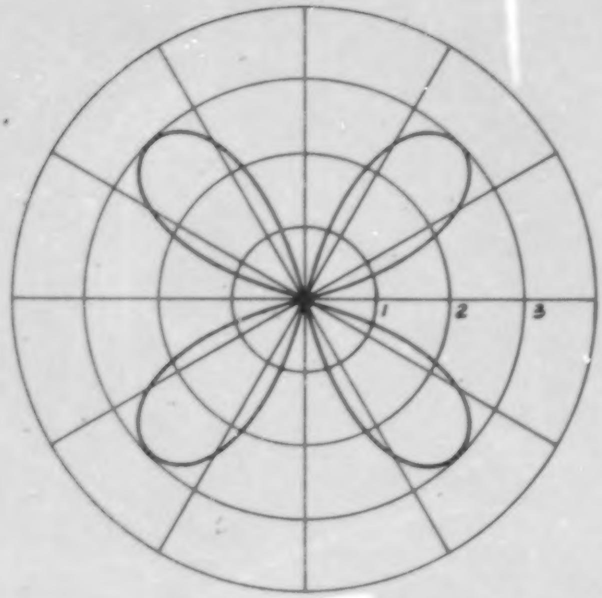


$r = 2 \sin 3\theta$

Three leafed rose symmetric with respect to $\theta = 30^\circ$, $\theta = 150^\circ$ and $\theta = 270^\circ$.

This last line has also the equation $\theta = 90^\circ$.

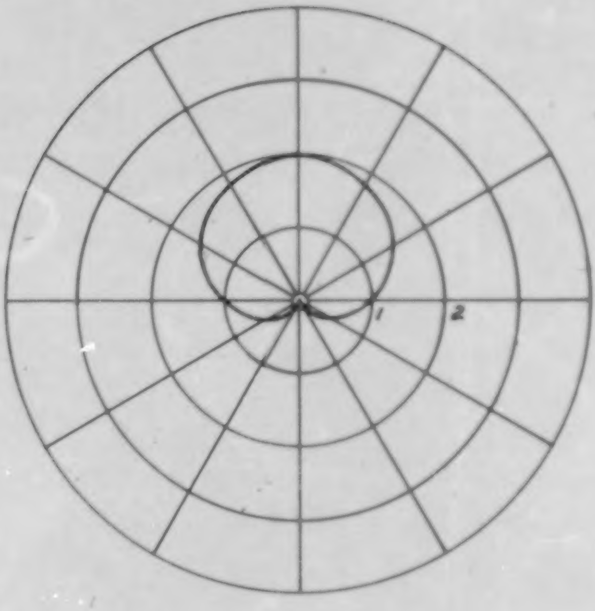
(e)



r = 3 sin 2 θ

Four leafed rose symmetric with respect to θ = 45°, θ = 135°, θ = 0° and θ = 90°. It is also symmetric with respect to the pole.

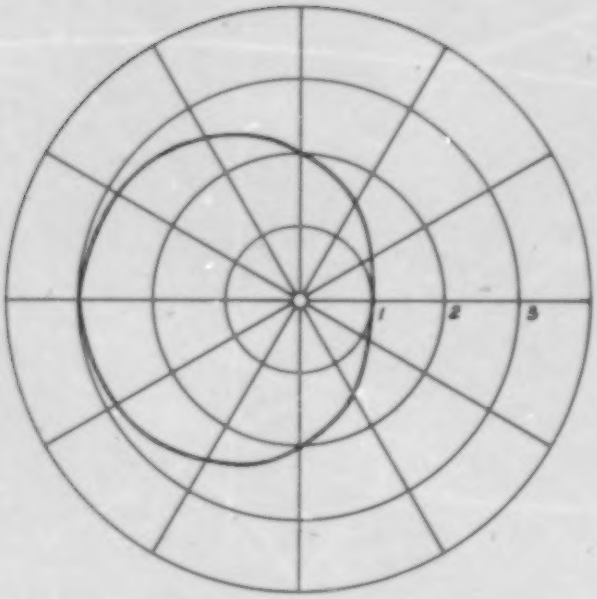
(f)



$r = 1 + \sin\theta$

Cardiod symmetric with respect to $\theta = 90^\circ$.

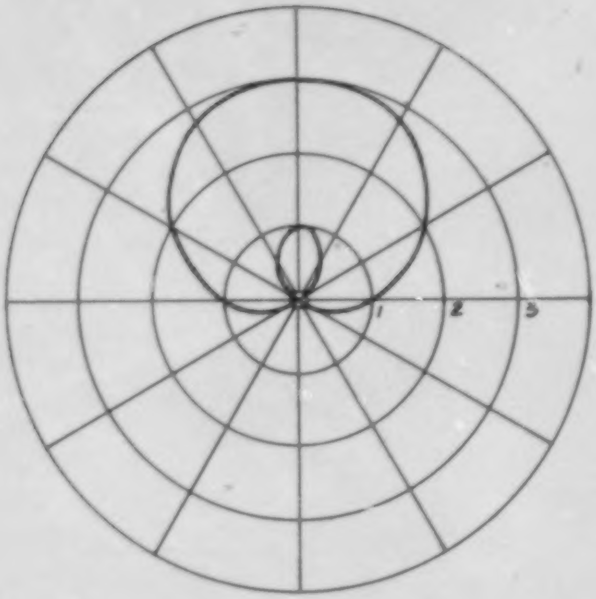
(g)



r = 2 - cos θ

Limacon symmetric with respect to $\theta = 0^\circ$.

(h)

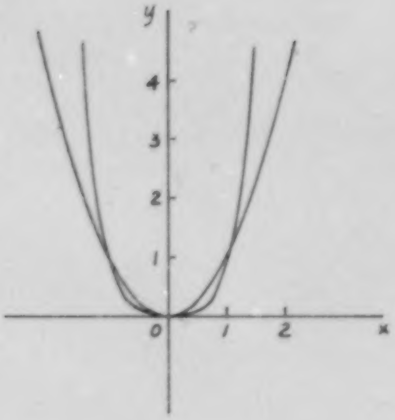


$r = 1 + 2 \sin\theta$

Limaçon symmetric with respect to $\theta = 90^\circ$.

Review Exercises

13. $y = x^2$ $y = x^4$

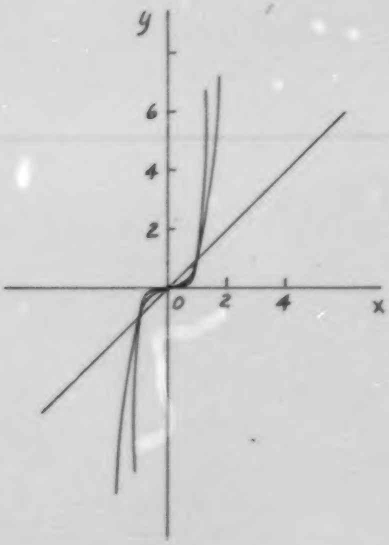


If $m > n$

for $|x| < 1$, $x^{2m} < x^{2n}$

for $|x| > 1$, $x^{2m} > x^{2n}$

14. $y = x$ $y = x^3$ $y = x^5$



Generalization.

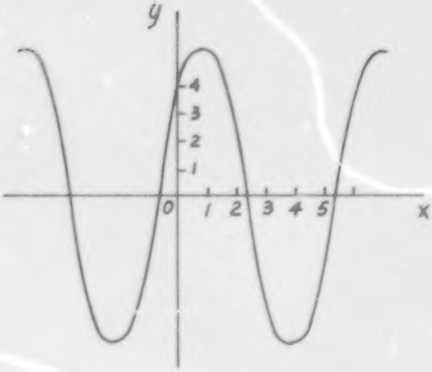
Let m , n be odd integers,

$0 < n < m$.

Then for

$x < -1$ we get	$x^m < x^n < -1$
$x = -1$	$x^m = x^n = -1$
$-1 < x < 0$	$-1 < x^n < x^m < 0$
$x = 0$	$x^n = x^m = 0$
$0 < x < 1$	$0 < x^m < x^n < 1$
$x = 1$	$x^m = x^n = 0$
$1 < x$	$1 < x^n < x^m$

15. $y = 3 \sin x + 4 \cos x$ $\qquad\qquad$ $y = 5(\frac{3}{5} \sin x + \frac{4}{5} \cos x)$



We know that $\sin(\alpha + \beta) = \sin\alpha \cos\beta + \sin\beta \cos\alpha$.

Let $\arccos \frac{3}{5} = \theta \approx 53^\circ$.

Then $\sin(x + \theta) = \frac{3}{5} \sin x + \frac{4}{5} \cos x$.

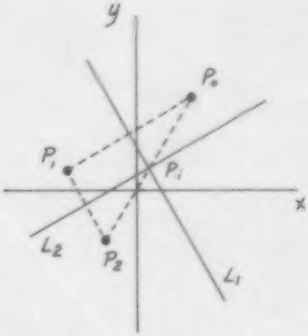
So $y = 5 \sin(x + \theta) \approx 5 \sin(x + 53^\circ)$.

16. If $y = a \sin x + b \cos x$ we may write

$$y = \sqrt{a^2 + b^2}\left(\frac{a}{\sqrt{a^2 + b^2}} \sin x + \frac{b}{\sqrt{a^2 + b^2}} \cos x\right)$$

or letting $\qquad = \arccos \frac{a}{\sqrt{a^2 + b^2}}$

$$y = \sqrt{a^2 + b^2} \sin(x + \theta)$$



17. Let $L_1 : ax + by + c = 0$ and $L_2 : -bx + ay + d = 0$ be two perpendicular lines. Let S be a set symmetric with L_1 and L_2.

Denote the intersection of L_1 and L_2 by P_i.

$$P_i = \left(\frac{bd - ac}{a^2 + b^2}, \frac{-ad - bc}{a^2 + b^2}\right)$$

We need to show that for any P_0 in S, its reflection in P_i is also in S.

If $P_0 = (p_0,q_0)$ is in S, then the reflection $P_1 = (p_1,q_1)$ of P_0 in L_1 is still in S. Since $L_1 \perp L_2$, the point P_1 is determined by equations

$$(1) \qquad \frac{ap_1 + bq_1 + c}{\sqrt{a^2 + b^2}} = - \frac{ap_0 + bq_0 + c}{\sqrt{a^2 + b^2}}$$

$$(2) \qquad \frac{-bp_1 + aq_1 + d}{\sqrt{a^2 + b^2}} = \frac{-bp_0 + ap_0 + d}{\sqrt{a^2 + b^2}}$$

i.e., the conditions, $d(P_0,L_1) = d(P_1,L_1)$ and P_1, P_0 on opposite sides,
and $d(P_0,L_2) = d(P_1,L_2)$ and P_1, P_0 on same side.

Solving these we find

$$P_1 = (p_1,q_1) = \left(\frac{p_0(b^2 - a^2) - 2ac}{a^2 + b^2}, \frac{q_0(a^2 + b^2) - 2bc}{a^2 + b^2}\right)$$

Since P_1 is in S, the reflection P_2 of P_1 in L_2 is in S.

Setting up and solving equations analogous to (1) and (2) we find

$$P_2 = (p_2,q_2) = \left(2\frac{(bd - ac)}{a^2 + b^2} - p_0, 2\frac{-bc - ad}{a^2 + b^2} - q_0\right).$$

Recalling some theorems of earlier chapters, we see that P_i is the midpoint of $\overline{P_0P_2}$, hence that the reflection of P_0 in P_i is P_2. Thus for any P in S its reflection in P_i is the reflection in L_2 of the reflection of P in L_1, and is therefore in S.

18. Let $A = \{(x,y) : \text{for some } t,\ x = at + b \text{ and } y = f(t)\}$; the graph of the parametric equations $x = at + b$, $y = f(t)$.

$$B = \{(x,y) : y = f(\frac{x-b}{a})\} ; \text{ the graph of } y = f(\frac{x-b}{a}) .$$

We wish to show $A = B$. Assume $a \neq 0$. Suppose $(p,q) \in A$. This is so if and only if there is a t such that

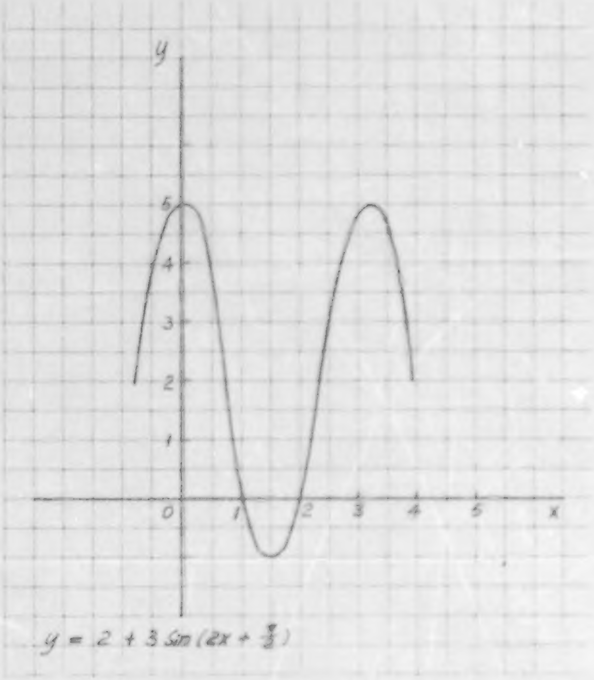
$$p = at + b \text{ and } q = f(t)$$

which is so if and only if

$$t = \frac{p-b}{a} \text{ and } q = f(\frac{p-b}{a})$$

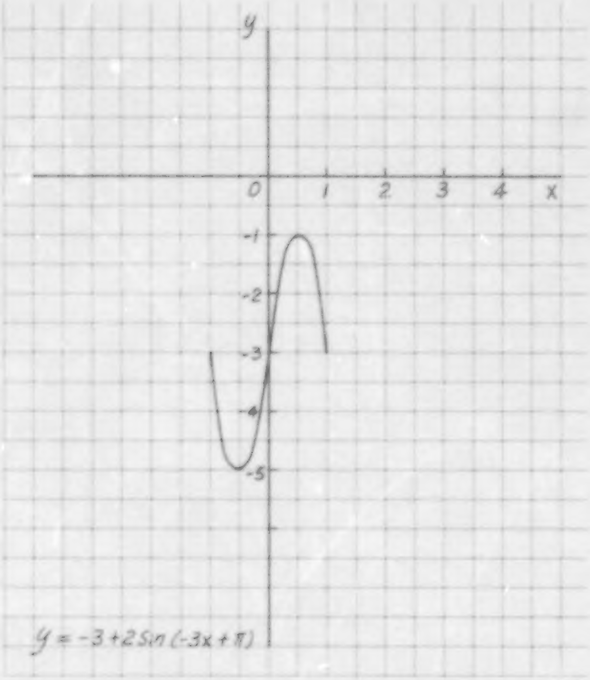
which is the defining condition that $(p,q) \in B$. Thus (p,q) A if and only if $(p,q) \in B$, i.e., $A = B$.

19. (a)

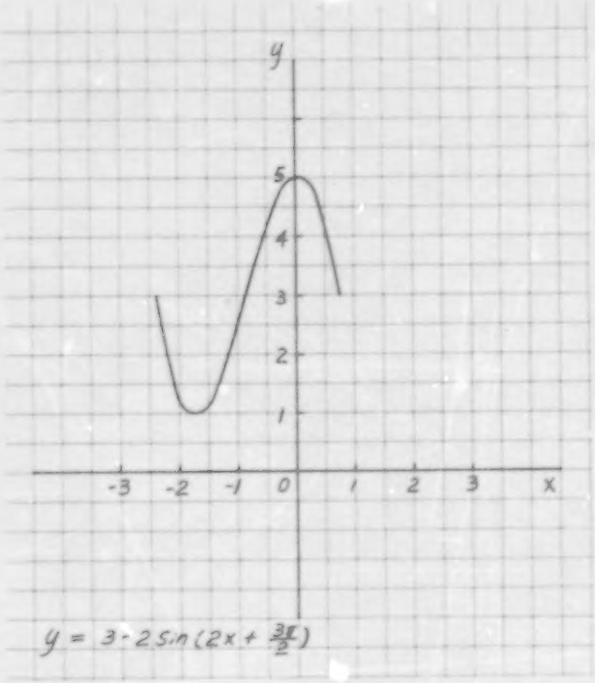


$y = 2 + 3\sin(2x + \frac{\pi}{3})$

(b)

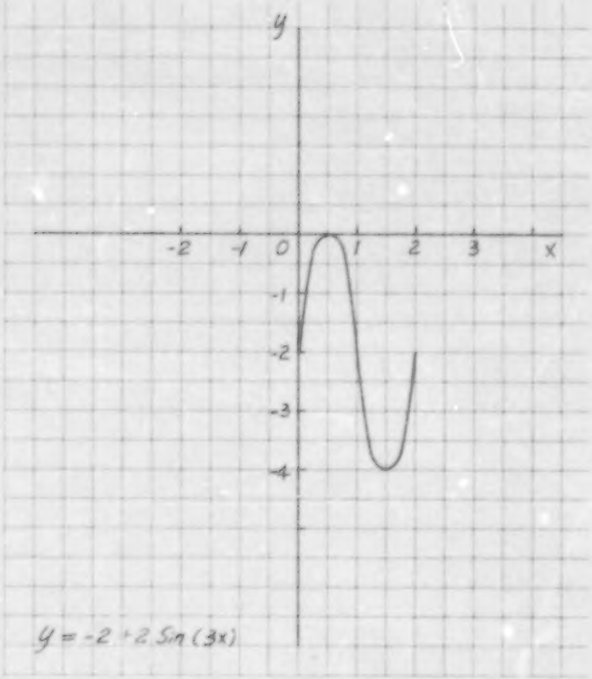


(c)



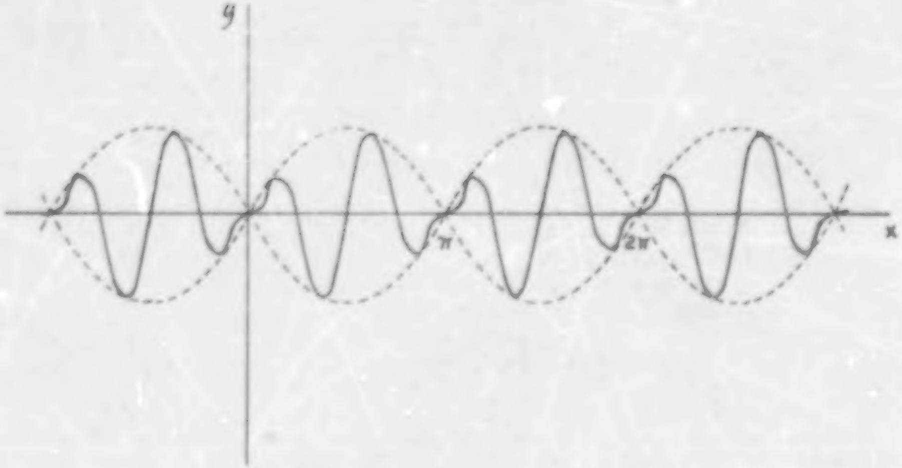
$y = 3 - 2\sin(2x + \frac{3\pi}{2})$

(d)

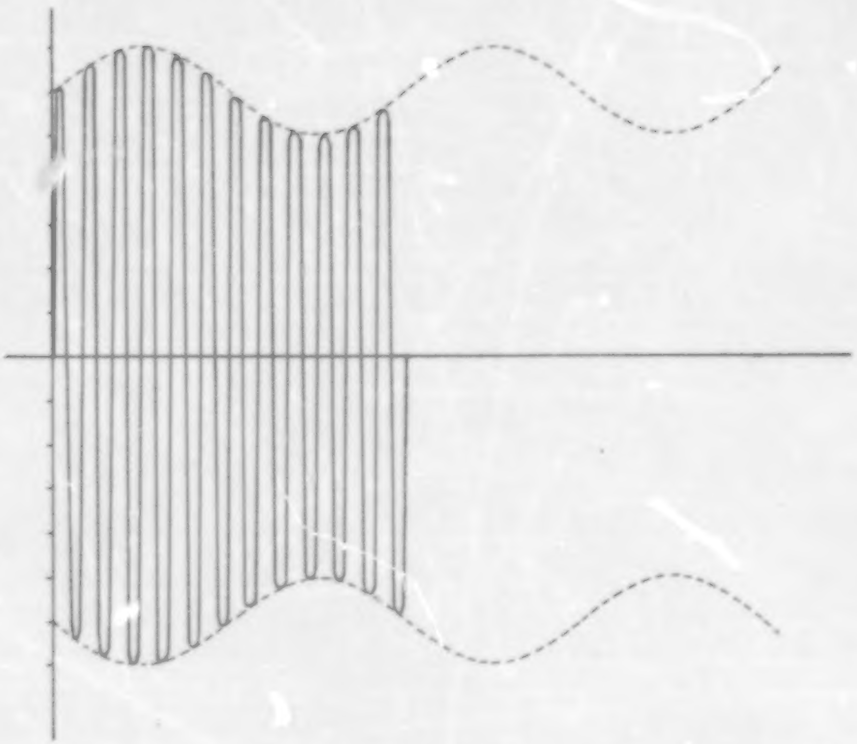


CHALLENGE EXERCISES

1. The graph of $y = \sin 4x \sin x$ may be thought of as a "rapidly" oscillating sine curve, $y_1 = \sin 4x$, modulated by a "slower" oscillating sine curve, $y_2 = \sin x$. Then, as in Example 8, the graph of $y = y_1 y_2$ will be constrained between the graphs of $y_2 = \sin x$ and $y_3 = \sin x$. The graph of y will touch the graph of y_2 whenever $y_1 = \pm 1$, that is, when $x = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \ldots$. The graph of y will cross the x axis when either y_1 or y_2 equals zero, that is, at $0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \ldots$. The graph looks like this (different scales on the axes):



The graph of $y = (6 + \sin x) \sin 12x$ is also a rapidly oscillating curve $y_2 = \sin 12x$ modulated by a slower oscillating curve $y_1 = 6 + \sin x$. The graph of y_1 is a sine curve elevated 6 units above the x-axis. It is bound between 5 and 7, therefore the graph of $y = y_1 y_2$ is constrained by symmetric curves $y_1 = 6 + \sin x$ and $y_3 = -6 - \sin x$ which bound a horizontal strip of periodically varying width, narrowest from 5 to -5 and widest from 7 to -7. The rest of the analysis of the graph is similar to that of the previous paragraph. The graph is drawn below (different scales along the axes).



The graph of $y = \sin(1000\,\pi t)\sin(1000\,000\,\pi t)$ will not be drawn but it may be analyzed in the same way as the others. We have a rapidly oscillating curve, $y_2 = \sin(1000000\pi t)$ modulated by a slower oscillating curve $y_1 = \sin(1000\pi t)$. Physicists would say that the first curve has a frequency of 500,000 cycles or 500 kilocycles or .5 megacycles per second. This is a reasonable radio frequency (RF). The second curve then has a frequency of 500 cps which is a reasonable audio frequency (AF). A further discussion of cycles and frequency would lead us too far into physics, and is left for further investigation there.

Teachers' Commentary

Chapter 7

CONIC SECTIONS

The student of Intermediate Mathematics has studied conic sections with equations given in simple form in rectangular coordinates; here we begin with something different. After taking up the introductory material in Section 7-1 and 7-2, if you feel that there is time, you may want to take next the first five sections of the Supplement to Chapter 7. In this you will find a careful development of the geometry of the plane sections of a right circular cone. This development relates the geometric properties of the conics to the cone, the cutting plane, and the sphere tangent to both of them. It is shown that, for a given conic section, the ratio of two cosines is a constant; this ratio, of course, is the eccentricity.

Section 7-3 develops equations in polar form for conics with focus at the pole, first with directrix perpendicular to the polar axis, and then with directrix parallel to the polar axis. (Cases in which there is rotation about the pole will be considered in Chapter 10.) The polar form emphasizes the essential similarity of the locus conditions for the ellipse, hyperbola, and parabola. Transformation of polar equations to familiar forms of the equations of the conics in rectangular coordinates is dealt with in Section 7-4. The exercises in both of these sections provide desirable review of polar coordinates.

The four positions of a conic considered in the text of Section 7-3, in Example 2, and in Exercises 9 and 10, give four forms of the equation which we summarize here.

$$r = \frac{ep}{1 \pm e \cos \theta} \qquad \begin{cases} \text{directrix} \perp \text{polar axis} \\ + \text{ if directrix contains } (p,0) \\ - \text{ if directrix contains } (-p,0) \end{cases}$$

$$r = \frac{ep}{1 \pm \sin \theta} \qquad \begin{cases} \text{directrix} \parallel \text{polar axis} \\ + \text{ if directrix contains } (p,\frac{\pi}{2}) \\ - \text{ if directrix contains } (-p,\frac{\pi}{2}) \end{cases}$$

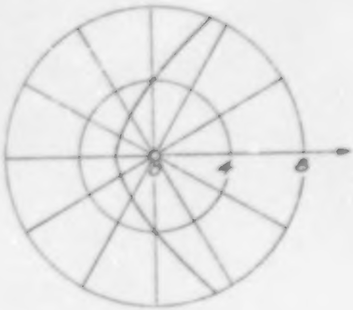
In all these cases the focus is at the pole.

The students should be urged, in doing Exercises 7-3, not to use point-by-point plotting alone. If they first rewrite the equations in a standard form, as indicated in Examples 3 and 4, they can tell what kind of conic section is represented. Then they should find intercepts, a few more points, and use symmetry.

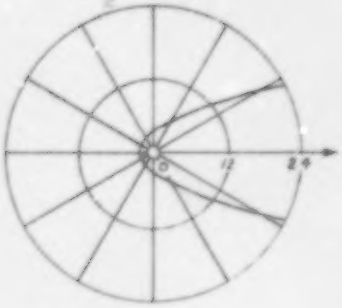
For the convenience of the teacher in making assignments, most of the exercises in this chapter are arranged so that even and odd exercises are roughly comparable. This does not include the applications toward the end of certain sets, or the challenge problems. In the case of exercises such as and 2 of Exercises 7-6, or 1 and 2 of the Review Exercises, with long lists of lettered parts, (a) , (c) , (e) , ... are comparable to (b), (d) , (f) , ...

Exercises 7-3

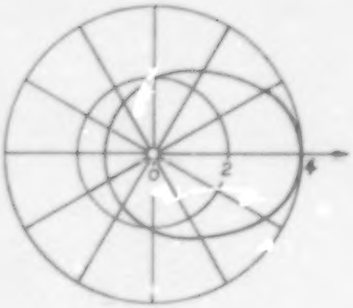
1. $r = \frac{4}{1 - \cos\theta}$



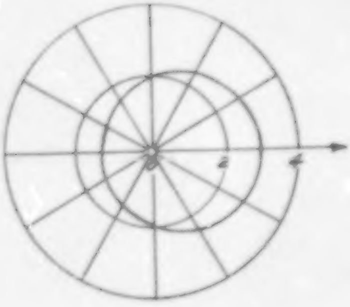
3. $r = \frac{2}{1 - \frac{1}{2}\cos\theta}$



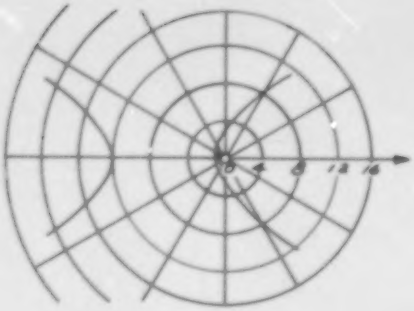
2. $r = \frac{3}{1 - \cos\theta}$



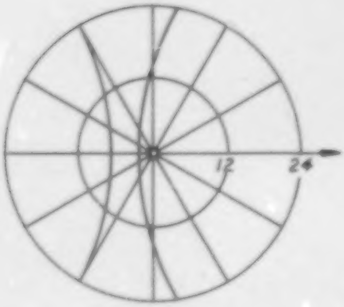
4. $r = \frac{2}{1 - \frac{1}{3}\cos\theta}$



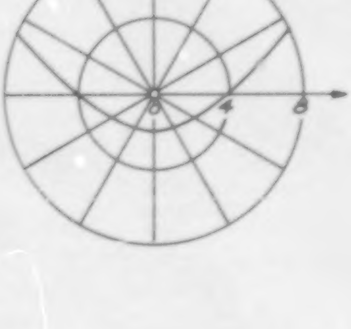
5. $r = \dfrac{3}{1 - \frac{5}{4}\cos\theta}$



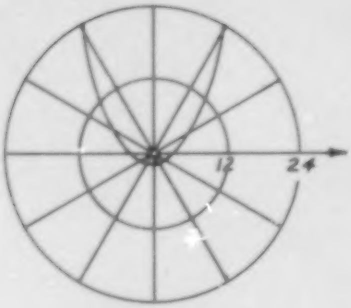
6. $r = \dfrac{12}{1 - 3\cos\theta}$



7. $r = \dfrac{4}{1 - \sin\theta}$



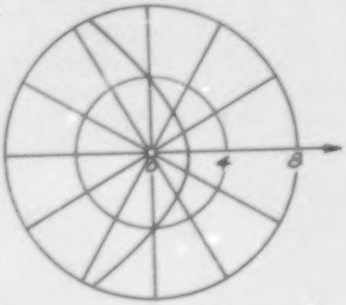
8. $r = \dfrac{3}{1 - \sin\theta}$



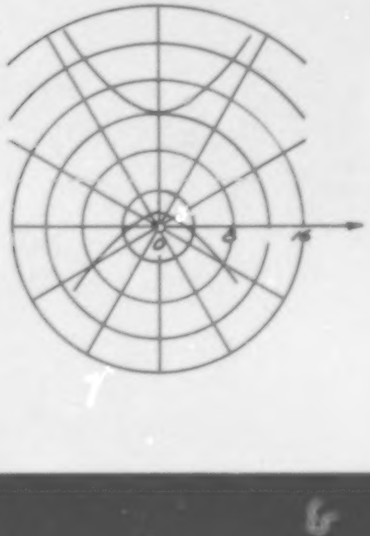
9. $r = \dfrac{ep}{1 + e\sin\theta}$

10. $r = \dfrac{ep}{1 + e\cos\theta}$

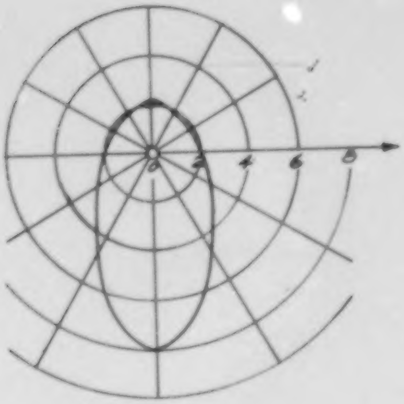
11. (a) $r = \dfrac{4}{1 + \cos\theta}$



(b) $r = \dfrac{3}{1 + \frac{5}{4}\sin\theta}$

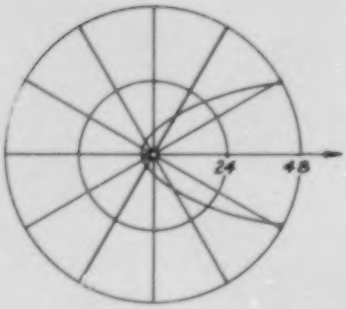


(c) $r = \dfrac{2}{1 + \frac{3}{4}\sin\theta}$

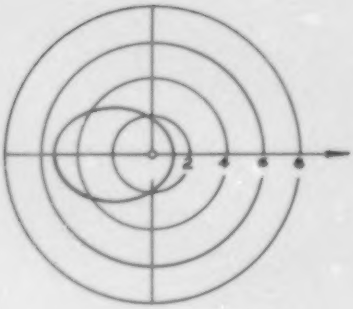


12. $r = \dfrac{6}{1 - \cos\theta}$

Parabola

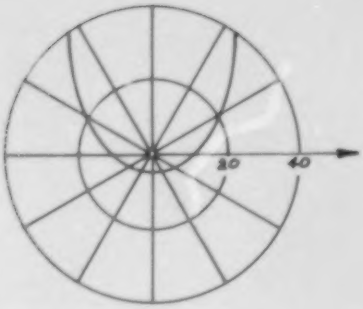


(d) $r = \dfrac{2}{1 + \frac{3}{5}\cos\theta}$



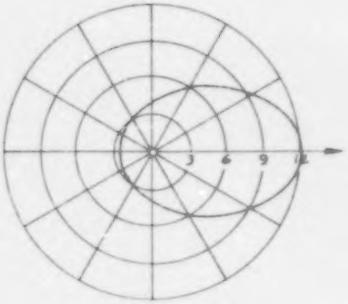
13. $r = \dfrac{10}{1 - \sin\theta}$

Parabola



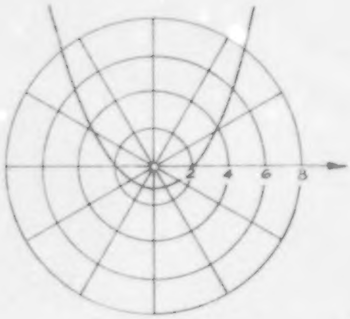
14. $r = \dfrac{4}{1 - \frac{2}{3}\cos\theta}$

Ellipse



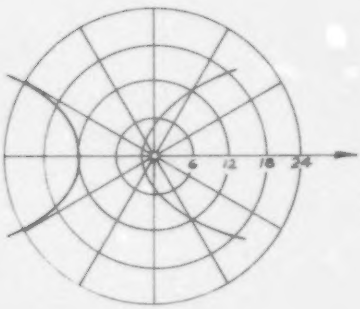
16. $r = \dfrac{2}{1 - \sin\theta}$

Parabola



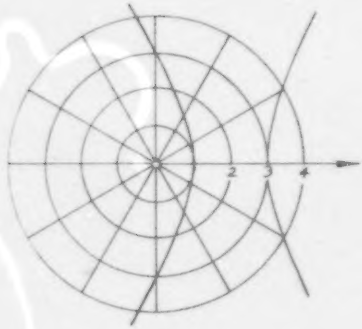
15. $r = \dfrac{4}{1 - \frac{4}{3}\cos\theta}$

Hyperbola



17. $r = \dfrac{3}{1 + 2\cos\theta}$

Hyperbola

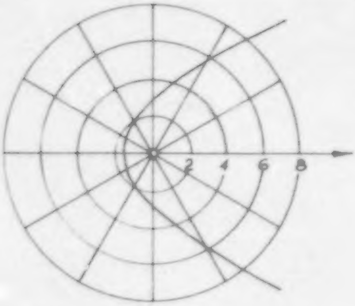


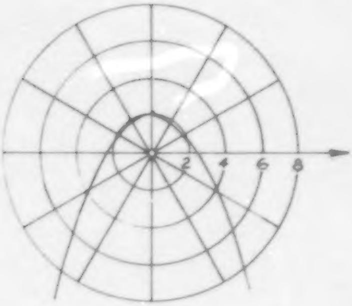
18. $r = \dfrac{3}{1 - \cos\theta}$

Parabola

19. $r = \dfrac{-2}{1 - \sin\theta}$

Parabola





20. The path is an ellipse with equation $r = \dfrac{ep}{1 - e\cos\theta}$. Two points of the path are given: $(5000, 180^\circ)$ and $(6000, 90^\circ)$. We substitute these coordinates in the equation and solve the resulting equations simultaneously to obtain $e = .2$, $p = 30,000$. The least distance is 5000 mi.; the greatest, when $\theta = 0^\circ$, is 7500 mi.

In Section 7-4 emphasis is on the algebra involved in transforming from the polar equations of the conic sections to the corresponding rectangular form. In the text we square both members of an equation to obtain Equation (2) . It is important that students understand why this is permissible; no doubt they have been warned that so doing may introduce points not in the graph of the original equation. The justification in the text depends on showing that squaring, in effect, introduces a new equation which is the related polar equation of the original equation and hence has the same graph. You might prefer a different proof, somewhat as follows.

Parabola: Since $x = r\cos\theta$, $\cos\theta = \frac{x}{r}$.

Then $$r = \frac{p}{1 - \cos\theta}$$

becomes $$r = \frac{p}{1 - \frac{x}{r}}$$

$$= \frac{rp}{r - x}\ .$$

This is the case if $$r - x = p$$

or $$r = x + p\ .$$

(It would appear that we have divided through by r at this point. This would mean that we would lose the solution $r = 0$. However, since $\cos\theta = \frac{x}{r}$, and consequently is not defined for $r = 0$, this value was not included in the original equation. In the following step we are in effect multiplying by an alternative equation with the same graph.)

We square both sides to obtain

$$r^2 = x^2 + 2px + p^2\ ,$$

and substitute for r^2 to obtain

$$x^2 + y^2 = x^2 + 2px + p^2$$

or

$$y^2 = 2p(x + \frac{p}{2})\ ,$$

which is a recognizable form for a parabola with vertex at $(-\frac{p}{2}, 0)$.

Ellipse: If

$$r = \frac{ep}{1 - e\cos\theta}\ ,\ \text{where}\ \ 0 < e < 1\ ,$$

then $$r = \frac{ep}{1 - \frac{ex}{r}}$$

or $$r = \frac{erp}{r - ex}\ .$$

This is the case if

$$r - ex = ep$$

or

$$r = e(x + p) .$$

(Once again we have removed the solution $r = 0$ which did not satisfy the original equation. In the following step we are again multiplying by an alternative equation with the same graph.)

We square both sides to obtain

$$r^2 = e^2(x^2 + 2px + p^2)$$

and substitute for r^2 to obtain

$$x^2 + y^2 = e^2(x^2 + 2px + p^2)$$

This is Equation (4) in the text.

We call your attention to the way in which directions for Exercise 7-4 have been written. Depending on what kind of practice your class needs, you would assign all of parts (a) , (b) , and (c) , or just the parts you wish to emphasize.

Exercise 7-4

The graphs are routine and will not be drawn.

1. $x^2 + y^2 = 9$, $r = -3$.
2. $x^2 + y^2 = 81$, $r = -9$.
3. $(x - 1)^2 + y^2 = 1$, related equation the same.
4. $x^2 + y^2 - x - y = 0$, related equation the same.
5. $y^2 = 16 + 8x$, $r = \frac{-4}{1 + \cos\theta}$
6. $y^2 = 9 - 6x$, $r = \frac{-3}{1 - \cos\theta}$
7. $\frac{(x + 2)^2}{1} - \frac{y^2}{3} = 3$, $r = \frac{-3}{1 + 2\cos\theta}$
8. $\frac{(x - 2)^2}{16} + \frac{y^2}{12} = 1$, $r = \frac{-6}{2 + \cos\theta}$

9. $\frac{(x - 2)^2}{9} + \frac{y^2}{5} = 1$, $r = \frac{-5}{3 + 2 \cos \theta}$

10. $\frac{(x + 3)^2}{4} - \frac{y^2}{5} = 1$, $r = \frac{-5}{2 + 3 \cos \theta}$

11. $(x^2 + y^2 - x)^2 = x^2 + y^2$ cardiod curve, $r = \cos \theta - 1$

12. $x^2 = 4 - 4y$, $r = \frac{-2}{1 + \sin \theta}$

13. $7x^2 + 16y^2 - 72x - 144 = 0$, $r = \frac{-12}{4 + 3 \cos \theta}$

14. $9y^2 - 16x^2 - 200y + 400 = 0$, $r = \frac{-20}{4 - 5 \sin \theta}$

Students of SMSG *Intermediate Mathematics* will have covered most of the material in Sections 7-5 through 7-8; it is in this text for convenience of reference. Ease in handling the simple forms of the equations of the conics is an important skill.

With able and well-prepared students, only a brief review of the text of these sections will be necessary. However, a number of the exercises should be done, both to reinforce previous learning and to develop further some of the properties and applications of conic sections. With such groups the teacher may want to take up the sections of the Supplement to Chapter 7 which deal with the general second-degree equation.

If the students are not familiar with the equations and basic properties of the conic sections summarized in the first paragraphs of these sections, the teacher will want to take time to develop this material with the class. *Intermediate Mathematics* would be helpful here.

While the amount of time that should be devoted to these sections will vary greatly with the training and ability of the class, it is urged that the time be sufficient for the students to develop some facility both with use of the locus definitions and with the standard forms.

The answers for Exercises 7-5 through 7-8 do not, in most cases, include the sketches the students are asked to make. However, use of the listed information about the curves will make it easy to check the students' sketches.

Exercises 7-5

1. Equation	Vertex	Focus	Axis	Directrix
(a) $x^2 = 4(-4)y$	$(0,0)$	$(0,-4)$	$x = 0$	$y = 4$
(b) $y^2 = 4(4)x$	$(0,0)$	$(4,0)$	$y = 0$	$x = -4$
(c) $x^2 = 4(\frac{3}{20})y$	$(0,0)$	$(0,\frac{3}{20})$	$x = 0$	$y = -\frac{3}{20}$
(d) $(y - \frac{5}{2})^2 = 4(-\frac{3}{2})(x - \frac{89}{24})$	$(\frac{89}{24},\frac{5}{2})$	$(\frac{53}{24},\frac{5}{2})$	$y = \frac{5}{2}$	$x = \frac{125}{24}$
(e) $(x - 2)^2 = 4(\frac{3}{8})(y - 1)$	$(2,1)$	$(2,\frac{11}{8})$	$x = 2$	$y = \frac{5}{8}$
(f) $(x+\frac{b}{2a})^2=4(\frac{1}{4a})(y-\frac{4ac-b^2}{4a})$	$(-\frac{b}{2a},\frac{4ac-b^2}{4a})$	$(-\frac{b}{2a},\frac{4ac-b^2+1}{4a})$	$x = -\frac{b}{2a}$	$y = \frac{4ac-b^2-1}{4a}$

2. (a) Case (a). Equation $x^2 = 0$. The y-axis.

Case (b). Equation: $(x - h)^2 = 0$. A line parallel to the y-axis.

Case (c). Equation: $y^2 = 0$. The x-axis.

(b) These cases occur when the cutting plane contains an element of the cone.

3. (a) $(y - 2)^2 = -6(x - \frac{1}{2})$

(b) $(x + 1)^2 = 2(y - \frac{5}{2})$

(c) $y^2 = -20x$

(d) $(y - 5)^2 = 4(x - 4)$

4. Same answers as for Problem 3.

5. (a) $y^2 = -10x$

(b) $(x - 2)^2 = -12(y + 3)$

(c) $y^2 = \frac{49}{2}x$

(d) $(x + 2)^2 = -16(y - 3)$

6. The equally spaced rulings permit locating points that are equally distant from a fixed line (L_0) and a fixed point (F). Thus P_2 is two units from L_0 (since it is on the second ruling away from L_0) and it is two units from F (since the radius used to determine it was two units, with F as center).

7. For every position of the pencil point P, the distance from P to the fixed line (L) is equal to the distance from P to the fixed point F.

Challenge Problems

1. The focus of the parabola is $F = (0, \frac{1}{4})$: the slope of the line containing P and F is $\frac{4a^2 - 1}{4a}$. Using this and the slope of the tangent line $(2a)$, we find that the tangent of the angle these lines form is $\frac{1}{2a}$. To avoid the problem presented by a vertical line, we use the fact that the angle between the tangent line and the parallel to the axis of the parabola is the complement of the angle formed by the tangent line and the x-axis; tangent of this last angle is $2a$; hence the tangent of its complement is $\frac{1}{2a}$.

2. (a) The tangent perpendicular to the line $y = 2ax - a^2$ must have slope $-\frac{1}{2a}$; therefore its point of contact is $P' = (-\frac{1}{4a}, \frac{1}{16a^2})$

   A test for collinearity can then be applied to the coordinates of P, P', and V.

   (b) Using the previous results, we obtain the equation of the tangent at P': $8ax + 16a^2y + 1 = 0$. We apply a test of concurrency to this equation, the equation of the tangent at P, and the equation of the directrix $(4y + 1 = 0)$.

Example 2 of Section 7-6 will give an opportunity to review with the students the technique of completing the square. Here the coefficients are numerical; when the method was used in Section 7-4 the coefficients were literal. Since the technique will continue to be useful here and elsewhere, we recommend that the teacher check that the students have facility with it. They should be able to handle not just the simplest cases (like the first ones in the exercise set), but also ones like 1(g), (h) and 5(c) of this set, and 3(g), (h) of Exercises 7-7.

Exercises 7-6

1. (a) $(x - 4)^2 + y^2 = 16$ $C = (4,0)$ $r = 4$
   (b) $(x - 3)^2 + (y - 5)^2 = 1$ $C = (3,5)$ $r = 1$
   (c) $(x - 2)^2 + (y + 4)^2 = 0$ Locus is the point $(2,-4)$.
   (d) $(x + 7)^2 + (y - \frac{9}{2})^2 = \frac{37}{4}$ $C = (-7,\frac{9}{2})$ $r = \frac{1}{2}\sqrt{37}$
   (e) $(x - \frac{1}{2})^2 + (y + \text{[illegible]})^2 = \frac{1}{2}$ $C = (\frac{1}{2}, -\frac{1}{2})$ $r = \frac{1}{2}\sqrt{2}$
   (f) $(x - a)^2 + (y - b)^2 = 0$ Locus is the point (a,b) .
   (g) $(x - \frac{3}{5})^2 + (y + \frac{2}{5})^2 = \frac{3}{25}$ $C = (\frac{3}{5}, -\frac{2}{5})$ $r = \frac{1}{5}\sqrt{3}$
   (h) $(x - \frac{a}{2})^2 + (y + \frac{b}{2})^2 = \frac{a^2 + 2ab + b^2}{4}$ $C = (\frac{a}{2}, -\frac{b}{2})$ $r = \frac{a + b}{2}$

2. (a) $x^2 + y^2 - 6x + 10y - 15 = 0$
   (b) $x^2 + y^2 + 10x - 24y = 0$
   (c) $x^2 + y^2 - 6x - 4y + 9 = 0$ and $x^2 + y^2 - 6x - 4y + 4 = 0$
   (d) The center is $(2,1)$ or $(-1,4)$.
   Equations: $x^2 + y^2 - 4x - 2y - 4 = 0$ and $x^2 + y^2 + 2x - 8y + 8 = 0$.
   (e) $r = \frac{17}{5}$. Equation: $25x^2 + 25y^2 - 50x - 100y - 164 = 0$
   (f) If equation is written $x^2 + y^2 + Dx + Ey + F = 0$, substitution of the coordinates gives equations $2D + 3E + F + 13 = 0$, $5D + E + F + 26 = 0$, $3E + F + 3 = 0$. Final equation: $x^2 + y^2 - 5x - y = 0$

3. (a) Slope of radius to $(3,-4)$ is $-\frac{4}{3}$; therefore slope of tangent is $\frac{3}{4}$. Equation: $3x - 4y - 25 = 0$

   (b) Proceding as in part (a), equation of tangent is $x_1x + y_1y = x_1^2 + y_1^2$. Since (x_1,y_1) is a point of the circle, $x_1^2 + y_1^2 = r^2$; thus the desired equation is $x_1x + y_1y = r^2$.

4. (a) Since the center $(0,0)$, the point $(3,7)$, and a point of contact of the tangent determines a right triangle, the Pythagorean Theorem can be used. Length of tangent $= \sqrt{58 - 25} = \sqrt{33}$.

   (b) [See part (a)] $C(-\frac{D}{2},-\frac{E}{2})$, $r = \frac{1}{2}\sqrt{D^2 + E^2 - 4F}$

$$t^2 = (x_1 + \frac{D}{2})^2 + (y_1 + \frac{E}{2})^2 - \frac{1}{4}(D^2 + E^2 - 4F)$$

$$= x_1^2 + y_1^2 + Dx_1 + Ey_1 + F$$

   (c) If $t^2 = 0$, the point (x_1,y_1) is a point of the circle. If $t^2 < 0$, the distance from the center to (x_1,y_1) is less than the radius; hence the point (x_1,y_1) is a point of the interior of the circle.

5. (a) $a(x^2 + y^2 - 10x - 2y - 35) + b(x^2 + y^2 + 4x - 6y - 49) = 0$

   (b) Substitution of $x = 0$, $y = -6$ in (a) gives $13a = -23b$. If we let $a = 23$ and $b = 13$, an equation is

$$5x^2 + 5y^2 - 141x + 16y - 84 = 0 .$$

   (c) The terms containing x in the equation of part (a) are

$$(a + b)x^2 + (-10a + 4b)x .$$

   Thus the x-coordinate of the center (which we know is -5) is $\frac{5a - 2b}{a + b}$. From this we find $10a = -3b$; we let $a = -3$, $b = 10$. Using these values, the equation is $7x^2 + 7y^2 + 70x - 54y - 385 = 0$.

6. Let the circles have equations $(x - h)^2 + (y - k)^2 = r^2$ and $(x - h_1)^2 + (y - k_1)^2 = r_1^2$. Then an equation of the radical axis is $(x - h)^2 + (y - k)^2 - r^2 - \left((x - h_1)^2 + (y - k_1)^2 - r_1^2\right) = 0$. For either circle, the square of the length of the tangent from the point (x,y) to the circle is the square of the distance from the point to the center, minus the square of the radius. The condition that these two lengths be equal is

$$(x - h)^2 + (y - k)^2 - r^2 = (x - h_1)^2 + (y - k_1)^2 - r_1^2$$

But this is exactly the condition shown above, that the point (x,y) is on the radical axis.

7. As shown in Problem 6, the required point must be on the radical axis of each pair of circles. We find equations of two of the radical axes (say $6x - y = -8$ and $4x - 3y = 11$) and solve; the point is $(-\frac{5}{2}, -7)$.

8. Using the circles with equations in Problem 6, slope of line of centers is $\frac{k_1 - k}{h_1 - h}$. From equation of radical axis (also Problem 6), slope is $-\frac{h_1 - h}{k_1 - k}$, the negative reciprocal of slope of line of centers. (If $h_1 = h$, first line is parallel to y-axis and second to x-axis, and also they are perpendicular; opposite case if $k_1 = k$.)

9. For the first circle, we have $C_1 = (-\frac{D_1}{2}, -\frac{E_1}{2})$, $r_1^2 = \frac{D_1^2 + E_1^2 - 4F_1}{4}$; for the second, $C_2 = (-\frac{D_2}{2}, -\frac{E_2}{2})$, $r_2^2 = \frac{D_2^2 + E_2^2 - 4F_2}{4}$. By the definition of orthogonal circles, $r_1^2 + r_2^2 = d(C_1, C_2)^2$; this condition is $\frac{D_1^2 + E_1^2 - 4F_1}{4} + \frac{D_2^2 + E_2^2 - 4F_2}{4} = (-\frac{D_1}{2} + \frac{D_2}{2})^2 + (-\frac{E_1}{2} + \frac{E_2}{2})^2$. When simplified, this is the desired condition.

10. Use the condition in Problem 11; in (b) both members of the equations must be divided by 2 before the condition applies.

11. (a) $k = -2$
    (b) $k = 48$ (Equations must be rewritten in proper form.)

Challenge Problems

1. Let the equations of the circles be

$$C_1 : x^2 + y^2 + D_1x + E_1y + F_1 = 0$$

$$C_2 : x^2 + y^2 + D_2x + E_2y + F_2 = 0$$

$$C_3 : x^2 + y^2 + D_3x + E_3y + F_3 = 0 .$$

Then equations of the common chords of C_1 and C_2, C_2 and C_3, and C_1 and C_3 are, respectively,

$$L_1 : (D_1 - D_2)x + (E_1 - E_2)y + F_1 - F_2 = 0$$

$$L_2 : (D_2 - D_3)x + (E_2 - E_3)y + F_2 - F_3 = 0$$

$$L_3 : (D_1 - D_3)x + (E_1 - E_3)y + F_1 - F_3 = 0$$

The family of lines through the intersection of L_1 and L_2 has as an equation

$$a\big((D_1 - D_2)x + (E_1 - E_2)y + F_1 - F_2\big) + b\big((D_2 - D_3)x + (E_2 - E_3)y + F_2 - F_3\big) = 0$$

For the values $a = 1$ and $b = 1$ this equation becomes

$$(D_1 - D_3)x + (E_1 - E_3)y + F_1 - F_3 = 0.$$

But this is exactly the equation we had for L_3 ; hence the lines are concurrent.

(Note: This is, of course, not the only way to make this proof. It is possible to assign coordinates to the vertices of the triangle, and find, in terms of these coordinates equations of the common chords and coordinates of their point of intersection.)

2. The proof given here for Challenge Problem 1 also holds here; so would any other that did not use the fact that in Problem 1 the circles intersect.

The student is asked to explain the variation in shape of the ellipse from the fact that $b = a\sqrt{1 - e^2}$. He should be able to see that the nearer the value of e is to zero, the closer $\sqrt{1 - e^2}$ is to 1; in such cases the minor axis differs very little from the major axis. But if values close to 1 (but less than 1) are selected for e, $\sqrt{1 - e^2}$ can be made as small as one wishes, and hence the minor axis can be made small as compared with the major axis.

Exercises 7-7

1. $\frac{(x - 3)^2}{36} + \frac{(y - 2)^2}{16} = 1$; $F(3 + 2\sqrt{5},2)$, $F'(3 - 2\sqrt{5},2)$, $V(9,2)$, $V'(-3,2)$; $x = 3 \pm \frac{18}{5}\sqrt{5}$: $e = \frac{\sqrt{5}}{3}$

2. $\frac{x^2}{9} + \frac{y^2}{5} = 1$

3.

	Equation	e	F,F'	V,V'	Directrices
(a)	$\frac{x^2}{1^2} + \frac{y^2}{2^2} = 1$	$\frac{\sqrt{3}}{2}$	$(0,\pm\sqrt{3})$	$(0,2),(0,-2)$	$y = \pm\frac{4}{3}\sqrt{3}$
(b)	$\frac{x^2}{5^2} + \frac{y^2}{2^2} = 1$	$\frac{\sqrt{21}}{5}$	$(\pm\sqrt{21},0)$	$(\pm 5,0)$	$x = \pm\frac{25}{21}\sqrt{21}$
(c)	$\frac{x^2}{(\sqrt{2})^2} + \frac{y^2}{(\sqrt{3})^2} = 1$	$\frac{\sqrt{3}}{3}$	$(0,\pm 1)$	$(0,\pm\sqrt{3})$	$y = \pm 3$
(d)	$\frac{x^2}{(\frac{1}{2})^2} + \frac{y^2}{(\frac{1}{3})^2} = 1$	$\frac{\sqrt{5}}{3}$	$(\pm\frac{\sqrt{5}}{6},0)$	$(\pm\frac{1}{2},0)$	$x = \pm\frac{3}{10}\sqrt{5}$
(e)	$\frac{(x - 4)^2}{5^2} + \frac{(y + 3)^2}{6^2} = 1$	$\frac{\sqrt{11}}{6}$	$(4,-3 \pm \sqrt{11})$	$(4,3),(4,-9)$	$y = -3 \pm \frac{36}{11}\sqrt{11}$
(f)	$\frac{(x + 5)^2}{3^2} + \frac{(y + 1)^2}{2^2} = 1$	$\frac{\sqrt{5}}{3}$	$(-5 \pm \sqrt{5},-1)$	$(-2,-1),(-8,-1)$	$x = -5 \pm \frac{9}{5}\sqrt{5}$
(g)	$\frac{(x - 2)^2}{2^2} + \frac{y^2}{3^2} = 1$	$\frac{\sqrt{5}}{3}$	$(2,\pm\sqrt{5})$	$(2,3),(2,-3)$	$y = \pm\frac{9}{5}\sqrt{5}$
(h)	$\frac{(x + 1)^2}{2^2} + \frac{(y - 5)^2}{4^2} = 1$	$\frac{\sqrt{3}}{2}$	$(-1,5 \pm 2\sqrt{3})$	$(-1,9),(-1,1)$	$y = 5 \pm \frac{8}{3}\sqrt{3}$
(i)	$\frac{(x - 1)^2}{5^2} + \frac{(y + 3)^2}{4^2} = 0$	Locus is the point $(1,-3)$.			

4. (a) $\frac{x^2}{16} + \frac{y^2}{9} = 1$

(b) $\frac{x^2}{5} + \frac{y^2}{9} = 1$

(c) $\frac{(x - 3)^2}{25} + \frac{(y - 5)^2}{\frac{600}{49}} = 1$

(d) $\frac{(x - 1)^2}{16} + \frac{(y - 4)^2}{12} = 1$

5. The latus rectum of an ellipse is either of the two chords of an ellipse perpendicular to the major axis at a focus.

If in Equation (a) of Figure 7-4, we set $x = ae$, we find $y = \pm b\sqrt{1 - e^2} = \pm \frac{b^2}{a}$; thus, the length of a latus rectum is $\frac{2b^2}{a}$. (The same result is obtained in each of the other forms.)

6. For the ellipse with equation $\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1$,

$$\begin{aligned} r + r' &= \sqrt{(x - ae)^2 + y^2} + \sqrt{(x + ae)^2 + y^2} \\ &= \sqrt{(x - ae)^2 + (1 - e^2)(a^2 - x^2)} + \sqrt{(x + ae)^2 + (1 - e^2)(a^2 - x^2)} \\ &= \sqrt{a^2 - 2aex + e^2x^2} + \sqrt{a^2 + 2aex + e^2x^2} \\ &= a - ex + a + ex \\ &= 2a \end{aligned}$$

It should be noted that the first radical expression is equal to $a - ex$ rather than $ex - a$ because the largest possible x is a, and e is less than one; hence $a - ex$ is positive.

7. If $P(x,y)$ is any point on the ellipse, the fixed points are $F(c,0)$, $F'(-c,0)$, and the constant is $2a(a > c)$, then

(1) $$PF + PF' = 2a$$

$$\sqrt{(x - c)^2 + y^2} + \sqrt{(x + c)^2 + y^2} = 2a .$$

After eliminating radicals in the usual way, this becomes

$$(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2)$$

or, we let $$b^2 = a^2 - c^2 ,$$

$$b^2x^2 + a^2y^2 = a^2b^2$$

(2) or $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 .$$

This sketches the proof that if the coordinates of a point satisfy (1), they satisfy (2) [illegible] converse, we retrace our steps, but must use both signs when the square root is taken, so that there are four equations,

$$\pm\sqrt{(x - c)^2 + y^2} \pm \sqrt{(x + c)^2 + y^2} = 2a .$$

It can easily be shown, because of the requirement that $a > c$ and the fact that the two radicals represent two sides of a triangle of which the third side has length $2c$, that only the positive signs can be used.

8. Each point is located so that the sum of its distances from the fixed points (F,F') is a constant (the length of $\overline{VV'}$, greater than the length of FF').

9. See Problem 8. As the distance between the tacks increases, the ellipse becomes more elongated; as it decreases, the ellipse becomes more like a circle.

10. (a) $5x^2 + 9y^2 - 40x - 54y + 116 = 0$

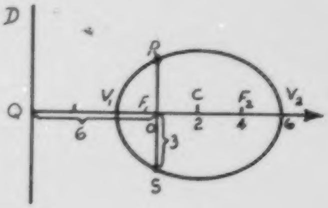
    (b) $8x^2 - 4xy + 5y^2 - 38x - 58y + 242 = 0$

11. $e = 0$; the focus-directrix definition cannot be used for a circle.

12. The ellipse has equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and is symmetric with respect to the origin and to both of the coordinate axes. Therefore $(-c,0)$ and the line $x = -\frac{a}{e} = -\frac{c}{e^2}$ are also a focus and directrix of the ellipse. (See Figure 7-4 part (a)).

13. We may recognize $r = \dfrac{6}{2 - \cos\theta}$ as an equation of an ellipse, and rewrite the equation in rectangular form for the purpose of discussion as in Example 1. Instead, we shall carry out the discussion in polar coordinates in order to illustrate the procedure in that system. We may rewrite the equation as

$$r = \frac{\frac{1}{2}(6)}{1 - \frac{1}{2}\cos\theta}$$



and see that the graph is an ellipse with one focus at the pole, with eccentricity $e = \frac{1}{2}$, and with directrix six units to the left of the pole and perpendicular to the line along the polar axis. From the definition of eccentricity we have

$$e = \frac{1}{2} = \frac{d(F_1,V_1)}{d(V_1,Q)} = \frac{d(F_1,V_2)}{d(V_2,Q)} .$$

Since $d(F_1,Q) = 6$, we have $d(F_1,V_1) = 2$, $d(F_1,V_2) = 6$. Therefore the vertices are $V_1 = (2,\pi)$ and $V_2 = (6,0)$. Since $d(F_2,V_2) = d(F_1,V_1) = 2$, we have the coordinates of the other focus, $F_2 = (4,0)$. Since the center of the ellipse is the midpoint of $\overline{F_1F_2}$, we have $C = (2,0)$. We readily find the major axis, $2a = d(V_1,V_2) = 8$, and the focal distance, $2c = d(F_1,F_2) = 4$. (We verify that $e = \frac{c}{a} = \frac{2}{4} = \frac{1}{2}$.) From the relationship $b^2 = a^2 - c^2$, we have $b^2 = 4^2 - 2^2 = 12$ and $b = 2\sqrt{3}$, which gives the minor axis, $2b = 4\sqrt{3}$. The length of a latus rectum (only one, RS, is drawn in the figure) can be found from the fact that it is twice the polar distance to the point R, for which $\theta = \frac{\pi}{2}$. Substitution in the original equation gives for this distance

$$d(F_1R) = \frac{6}{2 - \cos\frac{\pi}{2}} = 3 ;$$

therefore each latus rectum is of length 6. Using these values, we complete the sketch.

14. Using the representation $\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1$, the desired proportion is

$$2ae : 2a = 2a : 2\frac{a}{e} .$$

This can be verified immediately.

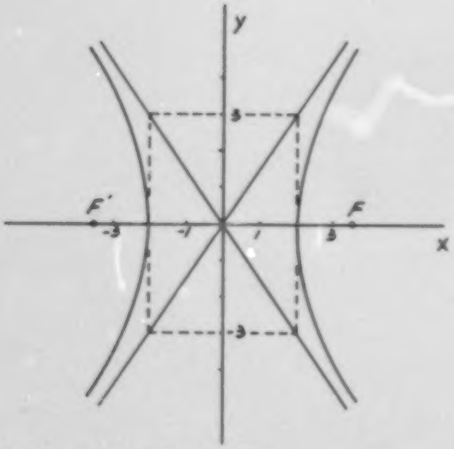
In the first four problems in Exercises 7-8, you will notice that the location of the transverse axis has been specified, but it has not been indicated which of the lengths given for the semi-axes is that of the transverse axis. This is deliberate, and it is suggested that you not make any additional specification in assigning the problems to the students. They should discover for themselves that two different hyperbolas meet the conditions in each problem, and should realize how this case differs from that of the ellipse, where the longer of the two axes must be the major axis.

Exercises 7-8

1. $\frac{x^2}{4} - \frac{y^2}{9} = 1$, $e = \frac{\sqrt{13}}{2}$, $V(2,0)$, $V'(-2,0)$, $F = (\sqrt{13},0)$, $F' = (-\sqrt{13},0)$

$D : x = \frac{4}{13}\sqrt{13}$, $D' : x = -\frac{4}{13}\sqrt{13}$

$A : y = \frac{3}{2}x$, $A' : y = -\frac{3}{2}x$



or

$\frac{x^2}{9} - \frac{y^2}{4} = 1$

$C = \frac{\sqrt{13}}{3}$, $V(3,0)$, $V'(-3,0)$, $F(\sqrt{13},0)$, $F'(-\sqrt{13},0)$

$D : x = \frac{9}{13}\sqrt{13}$, $D' : x = -\frac{9}{13}\sqrt{13}$

$A : y = \frac{2}{3}x$, $A' : y = -\frac{2}{3}x$

2. $-\frac{x^2}{4} + \frac{y^2}{9} = 1$

$e = \frac{\sqrt{13}}{3}$, $V(0,3)$, $V'(0,-3)$, $F(0,\sqrt{13})$, $F'(0,-\sqrt{13})$

$D : y = \frac{9}{13}\sqrt{13}$, $D' : y = -\frac{9}{13}\sqrt{13}$

$A : y = \frac{3}{2}x$, $A' : y = -\frac{3}{2}x$

or

$-\frac{x^2}{9} + \frac{y^2}{4} = 1$

$e = \frac{\sqrt{13}}{2}$, $V(0,2)$, $V'(0,-2)$, $F(0,\sqrt{13})$, $F'(0,-\sqrt{13})$

$D : y = \frac{4}{13}\sqrt{13}$, $D' : y = -\frac{4}{13}\sqrt{13}$

$A : y = \frac{2}{3}x$, $A' : y = -\frac{2}{3}x$

3. $\frac{(x + 2)^2}{16} - \frac{(y - 3)^2}{9} = 1$

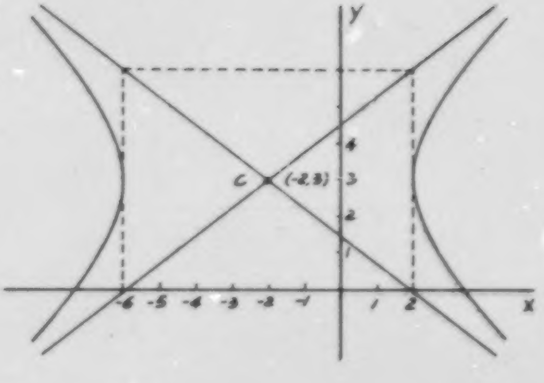
$e = \frac{5}{4}$, $V(2,3)$, $V'(-6,3)$

$F = (3,3)$, $(-7,3)$

$D : x = \frac{6}{5}$, $x = -\frac{26}{5}$

$A : 3x - 4y + 18 = 0$

$A' : 3x + 4y - 6 = 0$



or

$\frac{(x + 2)^2}{9} - \frac{(y - 3)^2}{16} = 1$

$e = \frac{5}{3}$, $V = (1,3)$, $V'(-5,3)$, $F(3,3)$, $F'(-7,3)$

$D : x = -\frac{1}{5}$, $D' : x = -\frac{19}{5}$

$A : 4x - 3y + 17 = 0$, $A' : 4x + 3y - 1 = 0$

4. $-\frac{(x+2)^2}{16}+\frac{(y-3)^2}{9}=1$

$e=\frac{5}{3}$, $V(-2,6)$, $V'(-2,0)$, $F(-2,8)$, $F'(-2,-2)$

$D : y=\frac{24}{5}$, $D' : y=\frac{6}{5}$

$A : 3x-4y+18=0$, $A' : 3x+4y-6=0$

or

$-\frac{(x+2)^2}{9}+\frac{(y-3)^2}{16}=1$

$e=\frac{5}{4}$, $V=(-2,7)$, $V'(-2,-1)$, $F=(-2,8)$, $F'(-2,-2)$

$D : y=\frac{31}{5}$, $D' : y=-\frac{1}{5}$

$A : 4x-3y+17=0$, $A' : 4x+3y-1=0$

5. See the next page.

6. (a) $-x^2+y^2=4$

(b) $x^2-y^2=4$

(c) $-4x^2+9y^2=36$

(d) $25x^2-144y^2=3600$

(e) $x^2-4y^2-4x+24y-48=0$

7. $16x^2-9y^2=144$

8. $2xy=1$; $e=\sqrt{2}$

9. It will be easier to do this proof if the coordinate system is chosen in such a fashion that the origin is the midpoint of the line segment determined by the two fixed points.

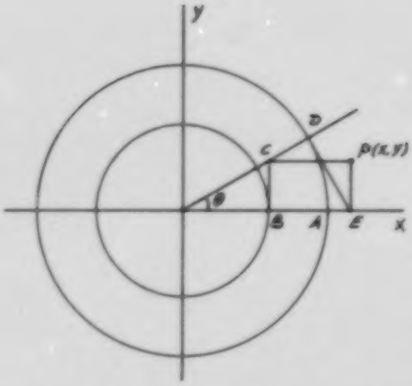
10. The latus rectum of a hyperbola is either of the two focal chords perpendicular to the transverse axis; its length is $\frac{2b^2}{a}$.

11. The points located by the construction lie on a hyperbola because the construction determines each one so that there is a constant difference $(2a)$ in its distances to the two fixed points.

12. Elimination of the parameter gives $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

5.	e	a	b	C	F	F'	V	V'	D,D'	A,A'
(a)	$\sqrt{2}$	2	2	(0,0)	$(2\sqrt{2},0)$	$(-2\sqrt{2},0)$	(2,0)	(-2,0)	$x = \pm\sqrt{2}$	$y = \pm x$
(b)	$\sqrt{2}$	2	2	(0,0)	$(0,2\sqrt{2})$	$(0,-2\sqrt{2})$	(0,2)	(0,-2)	$y = \pm\sqrt{2}$	$y = \pm x$
(c)	$\frac{\sqrt{13}}{3}$	3	2	(0,0)	$(\sqrt{13},0)$	$(-\sqrt{13},0)$	(3,0)	(-3,0)	$x = \pm \frac{9}{13}\sqrt{13}$	$y = \pm \frac{2}{3}x$
(d)	$\frac{13}{5}$	5	12	(0,0)	(0,13)	(0,-13)	(0,5)	(0,-5)	$y = \pm \frac{25}{13}$	$y = \pm \frac{5}{12}x$
(e)	$\sqrt{5}$	2	4	(2,3)	$(2,3 + 2\sqrt{5})$	$(2,3 - 2\sqrt{5})$	(2,5)	(2,1)	$y = 3 \pm \frac{2}{5}\sqrt{5}$	$x - 2y + 4 = 0, x + 2y - 8 = 0$

13. Draw concentric circles, center $(0,0)$, radii a and b, any angle θ. Draw tangent at B (intersecting OD at C) and tangent at D (intersecting x-axis at E). From C and E draw parallels to the x- and y-axes respectively, intersecting at $P(x,y)$. Then
$x = OE = a \sec \theta$ and
$y = CB = b \tan \theta$.



Hence $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \sec^2 \theta - \tan^2 \theta = 1$.

14. (a) $xy = -21$

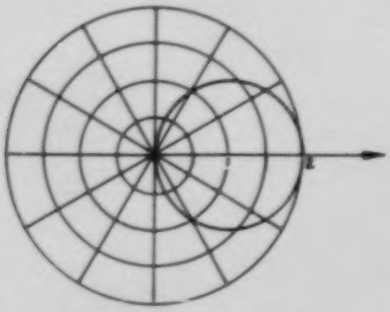
(b) $-x^2 + y^2 = 40$

15. Locus is a pair of intersecting lines (the asymptotes of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$).

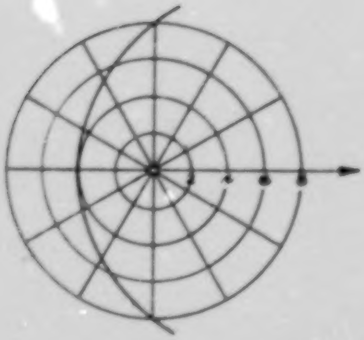
Review Exercises

1. (a) Circle with center at pole and radius of $\frac{2}{3}$.

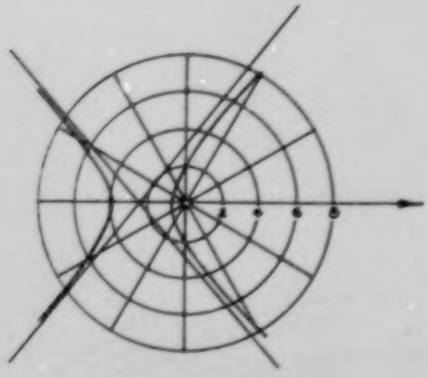
(b) Circle with center at $(1,0^\circ)$ and radius of 1.



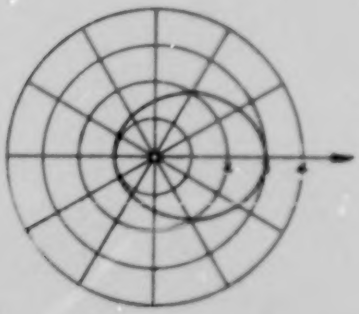
(c) Parabola with vertex at $(4,\pi)$, focus at 0 and directrix perpendicular to polar axis and 8 units to left of pole.



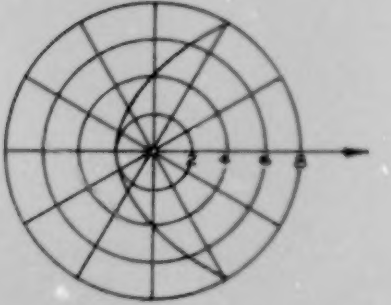
(d) Hyperbola with eccentricity of $\frac{3}{2}$, center at $(-\frac{12}{5}, 0^\circ)$, foci at $(0,0^\circ)$ and $(-\frac{24}{5}, 0^\circ)$, vertices at $(\frac{4}{5}, 180^\circ)$ and $(-4,0^\circ)$, and directrices $r \cos \theta = -\frac{4}{3}$ and $r \cos \theta = -\frac{52}{15}$.



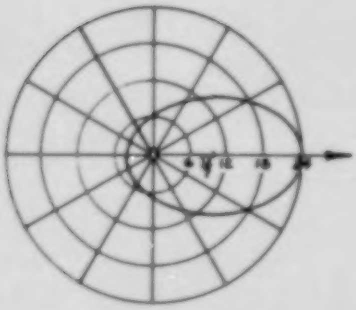
(e) Ellipse with eccentricity of $\frac{1}{2}$, center at $(1,0^\circ)$, foci at $(2,0^\circ)$ and $(0,0^\circ)$, directrices $r\cos\theta = 5$ and $r\cos\theta = -3$, vertices $(3,0^\circ)$ and $(1,180^\circ)$.



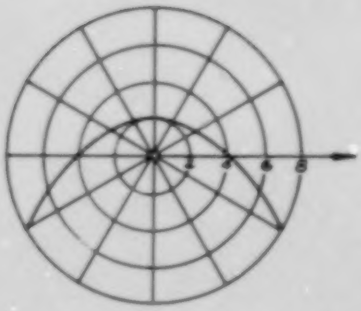
(f) Parabola with eccentricity of 1, focus at pole, vertex at $(2,180^\circ)$, and directrix $r\cos\theta = -4$.



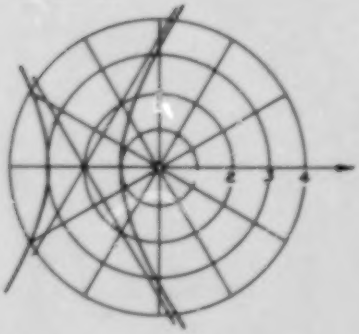
(g) Ellipse with eccentricity $\frac{3}{4}$, center at $(\frac{72}{7}, 0^\circ)$, foci at $(0,0^\circ)$ and $(\frac{144}{7}, 0^\circ)$, vertices at $(24,0^\circ)$ and $(\frac{24}{7}, 180^\circ)$, the length of the minor axis is $\frac{48}{\sqrt{7}}$, and the directrices are $r \cos \theta = \frac{200}{7}$ and $r \cos \theta = -8$.



(h) Parabola with eccentricity 0 , vertex $(2,90^\circ)$, focus at the pole, and directrix the line $r \sin \theta = 4$.



(i) Hyperbola with eccentricity 2, center at $(2,180^\circ)$, vertices at $(1,180^\circ)$ and $(-3,0^\circ)$, the foci are at $(4,180^\circ)$ and at the pole, and the directrices are $r \cos \theta = -\frac{3}{2}$ and $r \cos \theta = -\frac{5}{2}$.



(j) The graph is the point $(2,-3)$.

(k) Hyperbola with center at $(0,0)$, vertices at $(\sqrt{2},0)$ and $(-\sqrt{2},0)$, foci at $(\sqrt{5},0)$ and $(-\sqrt{5},0)$, eccentricity of $\sqrt{\frac{5}{2}}$, directrices $x = \frac{2}{\sqrt{5}}$ and $x = -\frac{2}{\sqrt{5}}$, and asymptotes $y = \sqrt{\frac{3}{2}}x$ and $y = -\sqrt{\frac{3}{2}}x$.

(ℓ) Parabola with vertex at $(-2,3)$, focus at $(-4,3)$ and directrix $x = 0$.

(m) Ellipse with center at $(-2,-4)$, vertices at $(-8,-4)$ and $(4,-4)$, foci at $(-2 \pm \sqrt{11}, -4)$, eccentricity of $\frac{\sqrt{11}}{6}$, and directrices $x = -2 \pm \frac{36}{\sqrt{11}}$.

(n) Ellipse with center at $(1,-2)$, vertices at $(1 \pm \sqrt{5}, -2)$, eccentricity of $\frac{\sqrt{10}}{5}$, foci at $(1 \pm \sqrt{2}, -2)$, and directrices $x = 1 + \frac{5}{\sqrt{2}}$ and $x = 1 - \frac{5}{\sqrt{2}}$.

(o) The graph is the point $(3,-5)$.

(p) Hyperbola with center at $(-4,-1)$, vertices at $(-4 \pm \sqrt{27}, -1)$, foci at $(2,-1)$ and $(-10,-1)$, eccentricity of $\frac{2}{\sqrt{3}}$, directrices $x = \frac{1}{2}$ and $x = -\frac{17}{2}$, and asymptotes $x + 4 \pm \sqrt{3}(y + 1) = 0$.

(q) Hyperbola with center at $(-2,3)$, vertices at $(3,3)$ and $(-7,3)$, eccentricity of $\frac{13}{5}$, foci at $(11,3)$ and $(-15,3)$, directrices $x = -\frac{1}{13}$ and $x = \frac{-51}{13}$, and asymptotes $y = \frac{12}{5}x + \frac{39}{5}$ and $y = -\frac{12}{5}x - \frac{9}{5}$.

2. (a) $y^2 = -20x$

(b) $(x - 7)^2 = 32(y - 6)$

(c) Four circles, centers $(\pm 5, \pm 5)$.

Equations: $x^2 + y^2 - 10x \pm 10y + 25 = 0$, $x^2 + y^2 + 10x \pm 10y + 25 = 0$.

(d) $r = 2\sqrt{2}$, Equation: $x^2 + y^2 - 2x + 8y + 9 = 0$

(e) $C = (0,4)$, $r = 2\sqrt{5}$. Equation: $x^2 + y^2 - 8y - 4 = 0$

(f) $x^2 + y^2 - 12x + 8y - 48 = 0$

(g) $\frac{(x - 2)^2}{9} + \frac{4(y - 3)^2}{27} = 1$

(h) $\frac{(x + 3)^2}{2} + \frac{(y - 3)^2}{6} = 1$

(i) $\frac{(x - 2)^2}{4} + \frac{(y - 1)^2}{5} = 1$

(j) $\frac{-9x^2}{319} + \frac{16y^2}{319} = 1$

(k) $2x^2 - x - y + 5 = 0$

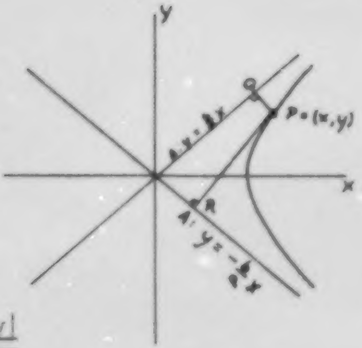
3. $(x + 1)^2 = 16y$

4. $(y + 1)^2 = -4(x - 2)$. Each center is equally distant from a fixed point $(1,-1)$ and a fixed line (the line $x = 3$).

5. $\frac{\sqrt{5}}{2}$

6. Elimination of the parameter gives $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

7. $3x^2 + 3y^2 + 6x - 14y + 6 = 0$.



8. Choose axes so that equation of curve is $b^2x^2 - a^2y^2 = a^2b^2$; then asymptotes are $y = \pm \frac{b}{a}x$. Then write expressions for distances from $P = (x,y)$ to asymptotes:

$$d(P,Q) \cdot d(P,R) = \frac{|bx - ay|}{\sqrt{a^2 + b^2}} \cdot \frac{|bx + ay|}{\sqrt{a^2 + b^2}}$$

$$= \frac{|b^2x^2 - a^2y^2|}{a^2 + b^2} = \frac{a^2b^2}{a^2 + b^2}$$

9. (a) If $\frac{b}{a} = 2$, $b = 2a$. Then $e = \frac{\sqrt{a^2 + b^2}}{a} = \frac{\sqrt{5a^2}}{a} = \sqrt{5}$

(b) In similar fashion, $e = \sqrt{1 + k^2}$

10. (a) Since $x^2 + y^2 = \frac{t^2r^2 + r^2}{1 + t^2}$, we get $x^2 + y^2 = r^2$.

(b) If only positive (or only negative) signs are used, the graph is only one-fourth of a circle; which part depends on the signs used, and also on whether r and t are positive or negative. If + signs are used, and r and t are both positive, it is the part in the first quadrant; if + signs are used, and $r > 0$, $t < 0$, it is the part in the second quadrant; and so on.

(c) In order for x to be $\pm r$, $\frac{t}{\sqrt{1 + t^2}} = \pm 1$. This is impossible, for $\sqrt{1 + t^2} = t$ only if $t = 0$, and then $x = 0$. Thus to be precise we would say that the parametric equations represent a circle with two points missing.

11. $e = \frac{\sqrt{a^2 + b^2}}{a}$, $e' = \frac{\sqrt{a^2 + b^2}}{b}$. $\frac{1}{e^2} + \frac{1}{e'^2} = \frac{a^2 + b^2}{a^2 + b^2} = 1$

12. $\frac{(x-a)^2}{k^2} + \frac{(y-b)^2}{k^2} = 1$. This is a circle with center (a,b) and radius $|k|$.

If k were zero, then the locus would be reduced to the point (a,b) .

13. Computed height at edge of road is $\frac{20\sqrt{5}}{3}$ ft. (≈ 14.9 ft.).

14. 20 feet

15. Let the equation of the hyperbola be $x^2 - y^2 = a^2$; then $e = \sqrt{2}$, $F = (ae,0)$, $F' = (-ae,0)$, and for a point (x,y) on the curve,

$$r = \sqrt{(x - ae)^2 + y^2} = \sqrt{(x - a\sqrt{2})^2 + x^2 - a^2} = \sqrt{(2x^2 + a^2) - 2a\sqrt{2}\,x}$$

similarly

$$r' = \sqrt{(2x^2 + a^2) + 2a\sqrt{2}\,x}$$

$$r \cdot r' = \sqrt{(2x^2 + a^2)^2 - 8a^2x^2}$$

$$= \sqrt{4x^4 - 4ax + a^4}$$

$$= |2x^2 - a^2| = 2x^2 - a^2 .$$

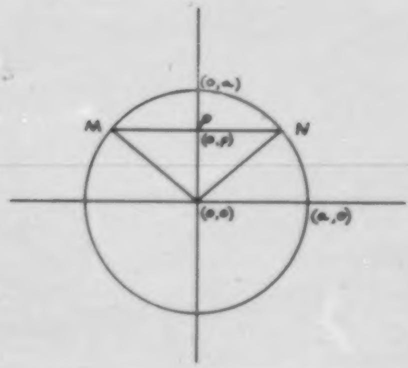
The square of the distance from the point to the center $(0,0)$ is $x^2 + y^2 = x^2 + x^2 - a^2 = 2x^2 - a^2$

16. (a) One possible form is $\frac{x^2}{a^2} + \frac{y^2}{\frac{16}{25}a^2} = 1$

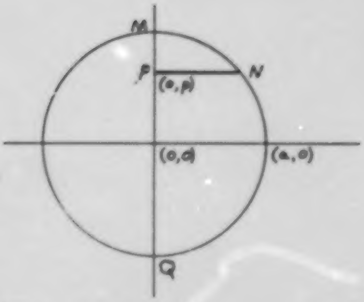
(b) $\frac{4x^2}{225} + \frac{y^2}{36} = 1$

(c) $\frac{x^2}{25} + \frac{y^2}{16} = 1$

17. (a) The equation of the circle is $x^2 + y^2 = a^2$ and the equation of the chord is $y = p$. If $y = p$, then $x^2 = a^2 - p^2$ or $x = \pm\sqrt{a^2 - p^2}$



Then $d(P,N) = \sqrt{a^2 - p^2}$ and $d(P,M) = \sqrt{a^2 - p^2}$.



(b) $x^2 + y^2 = a^2$

$d(P,N) = \sqrt{a^2 - p^2}$, $d(M,P) = a - p$,

and $d(P,Q) = a + p$.

Then $\frac{d(M,P)}{d(P,N)} = \frac{d(P,N)}{d(P,Q)}$.

(c) Let (0,0) be one point and $P = (p,q)$ be the other.

$$\sqrt{(x - p)^2 + (y - q)^2} = k^2\sqrt{x^2 + y^2}$$

$$(k^2 - 1)x^2 + (k^2 - 1)y^2 + 2px + 2qy = p^2 + q^2$$

$$(x + \frac{p}{k^2 - 1})^2 + (y + \frac{q}{k^2 - 1})^2 = \frac{k^2(p^2 + q^2)}{(k^2 - 1)^2}$$

This is the equation of a circle.

We must restrict k so that is positive and not equal to one.

Challenge Problems

1. Let the hyperbola have equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Then the equations of the three lines named are, in order, $y = \frac{b}{a}x$, $x = \frac{a}{e}$, $y = \frac{-a}{b}(x - ae)$. These lines can be proved concurrent in any of a variety of ways.

2. If $P = (x,y)$ is the point where the explosion takes place, the 5-second time difference at A and B gives the condition

$$\sqrt{x^2 + y^2} - \sqrt{(x - 2)^2 + y^2} = 5(.2) ,$$

which becomes $\frac{(x - 1)^2}{.25} - \frac{y^2}{.75} = 1$.

The 8-second difference at A and C gives the condition

$$\sqrt{x^2 + y^2} - \sqrt{x^2 + (y - 4)^2} = 8(.2) ,$$

which becomes $\frac{(y - 2)^2}{.64} - \frac{x^2}{3.36} = 1$.

If we write equations of the appropriate asymptotes (the ones we want have positive slope), we have $y = \sqrt{3}(x - 1)$ and

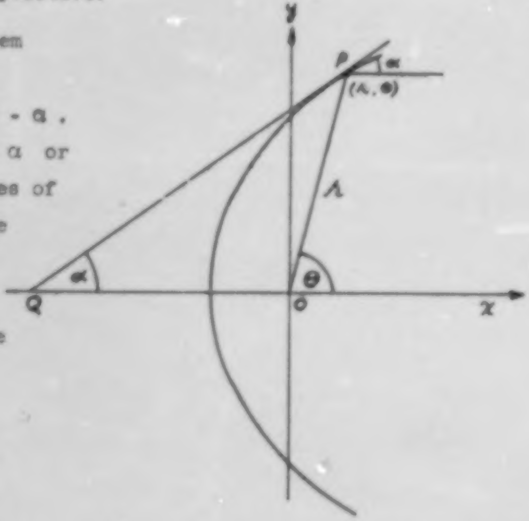
$y = \frac{2}{\sqrt{21}} x + 2$. Solving these equations simultaneously, we find that the point of intersection is approximately $(2.9, 3.3)$. While a point of intersection of the asymptotes is not a point of intersection of the curves, it is probably satisfactory here since there was only one significant figure in the times given.

3. If the suggestion is followed, the condition is

$$d(P,W_2) - d(P,W_1) \geq 20$$

which is $\sqrt{(x + 15)^2 + y^2} - \sqrt{(x - 15)^2 + y^2} \geq 20$.

This becomes $\frac{x^2}{100} - \frac{y^2}{125} \geq 1$. The locus has as its boundary the part of the hyperbola for which x is positive.



4. From the statement of the problem and the diagram, we must show $m \angle OPQ = \alpha$. But $m \angle OPQ = \theta - \alpha$. Therefore we must show $\alpha = \theta - \alpha$ or $\theta = 2\alpha$. Rectangular coordinates of P are $(r \cos \theta, r \sin \theta)$; the equation of the parabola in rectangular coordinates is $y = 12(x + 3)$. The point-slope form of the equation of a line through P with slope m is

$$y - r \sin \theta = m(x - r \cos \theta) ,$$

or

$$y = mx + r(\sin \theta - m \cos \theta) .$$

This line will in general intersect the parabola in two points, but if it is a tangent line there will be just one such point. The coordinates of the intersection points can be found by solving simultaneously the equations of the line and parabola.

Thus by substituting we get a single equation for the x-coordinate,

$$\big(mx + r(\sin \theta - m \cos \theta)\big)^2 = 12(\alpha + 3) ,$$

or

$$m^2x^2 + \Big(2mr(\sin\theta - m\cos\theta) - 12\Big)x + r^2(\sin\theta - m\cos\theta)^2 - 36 = 0 .$$

Tangency requires that the roots of this equation be equal; therefore, the discriminant of the equation must equal zero. Hence

$$\Big(2mr(\sin\theta - m\cos\theta) - 12\Big)^2 - 4m^2\Big(r^2(\sin\theta - m\cos\theta)^2 - 36\Big) = 0 .$$

This equation can be eventually simplified to

$$3m^3 - mr(\sin\theta - m\cos\theta) + 3 = 0 .$$

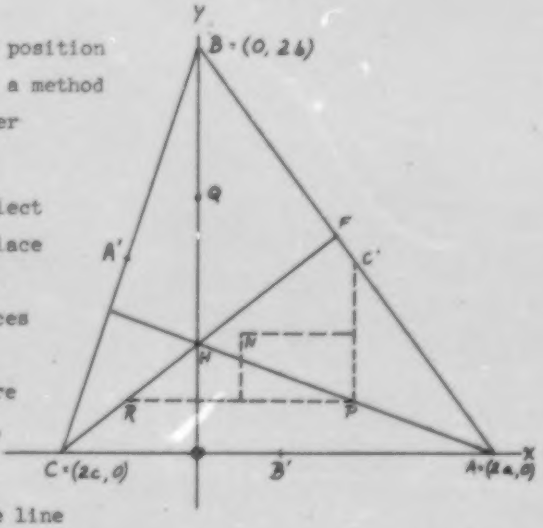
But for this parabola $r = \frac{6}{1 - \cos\theta}$; substituting this in the equation just above, we obtain, with some more simplification,

$$(1 + \cos\theta)m^2 - (2\sin\theta)m + 1 - \cos\theta = 0 .$$

Solving this for m gives the single value, $m = \frac{\sin\theta}{1 + \cos\theta}$.

But this is identically equal to $\tan\frac{\theta}{2}$. Since m is the tangent of the angle of inclination, $\angle$, we have $\alpha = \frac{\theta}{2}$ or $\theta = 2\alpha$, which is what we wanted to prove.

5. We indicate here one possible position of the triangle, and indicate a method of proof. There are many other possibilites.



In triangle ABC , we select one altitude as y-axis, and place the origin at the foot of the altitude. Then let the vertices be $A = (2a,0)$, $B = (0,2b)$, $C = (2c,0)$. The midpoints are $A' = (c,b)$, $B' = (a + c, 0)$, $C' = (a,b)$.

Altitude $\overline{CF}$ lies in the line with equation

$$y = \frac{a}{b}(x - 2c) .$$

It intersects altitude $\overline{BO}$ in point $H = (0, \frac{-2ac}{b})$. Hence the midpoints of $\overline{CH}$ and $\overline{AH}$ are $R = (c, \frac{-ac}{b})$ and $P = (a, -\frac{ac}{b})$.

The center of the circle through R, P, and C' would lie on the perpendicular bisectors of $\overline{RP}$ and $\overline{PC'}$; the point in which they intersect is

$$N = \left(\frac{a + c}{2}, \frac{b^2 - ac}{2b}\right).$$

Now we verify that the remaining six points (O, D, F, Q, B', A') lie on the same circle. One way would be to find the radius,

$$r = \frac{1}{2b}\sqrt{(a^2 + b^2)(b^2 + c^2)},$$

and verify that it is equal to the distance from N to each of these points.

Illustrative Test Items

1. Identify and sketch the curves whose equations are given.

   (a) $r - 5 = 0$

   (b) $r = 2 \sin \theta$

   (c) $r = \frac{3}{1 - \cos \theta}$

   (d) $r = \frac{4}{2 - \cos \theta}$

   (e) $r = \frac{6}{4 - 8 \cos \theta}$

   (f) $3y^2 - 4x^2 = 12$

   (g) $x^2 - 4y - 4 = 0$

   (h) $16x^2 + 25y^2 = 400$

   (i) $9x^2 + 4y^2 - 36x + 32y + 100 = 0$

   (j) $x^2 - 25y^2 + 2x + 100y - 99 = 0$

   (k) $16x^2 - 9y^2 + 32x + 54y - 209 = 0$

   (ℓ) $9x^2 + 4y^2 - 18x + 16y - 11 = 0$

2. Sketch the graphs of the following polar equations.
   Write the equations in rectangular form.

   (a) $2r - 7 = 0$

   (b) $r = \frac{3}{1 - 2 \cos \theta}$

   (c) $r - r \sin \theta - 2 = 0$

   (d) $r = \frac{5}{3 - 2 \cos \theta}$

3. Identify the following conic sections; give the eccentricity.

   (a) $r = \frac{6}{3 - \cos \theta}$

   (b) $2r - 3r \cos \theta - 12 = 0$

   (c) $2r - 5 = 0$

   (d) $r = 4 - r \cos \theta$

4. The directrix of a parabola is the line $y = x$, and the focus is $(4,-4)$. What are the coordinates of the vertex?

5. The eccentricity of a hyperbola is 2 and the distance between the foci is 8. Find the lengths of the semi-axes.

6. Write an equation of the tangent to the circle $x^2 + y^2 = 25$ at the point $(-3,4)$.

7. Find an equation of the radical axis of the circles with equations $(x - 3)^2 + (y + 2)^2 = 4$ and $x^2 + y^2 = 9$.

8. What kind of symmetry do the graphs of the following equations have? If there is point-symmetry, give the coordinates of the point; if line-symmetry, give an equation of each axis of symmetry.

   (a) $r = \frac{6}{3 - 3\cos\theta}$

   (b) $r = \cos\theta + \sin\theta$

   (c) $25x^2 - 4y^2 = 100$

   (d) $\frac{(x - 2)^2}{4} + \frac{(y + 4)^2}{36} = 1$

   (e) $\frac{(x + 3)^2}{25} - \frac{(y - 1)^2}{4} = 1$

   (f) $x^2 - 6x - y + 7 = 0$

9. Write an equation of a circle with center $(3,-1)$ and tangent to the line with the equation $2x + 5y - 5 = 0$.

10. The axes of an ellipse have lengths 10 and 6; what is its eccentricity?

11. Find the distance between the foci of the conic section with equation $\frac{x^2}{4} - \frac{y^2}{9} = 1$.

12. The vertex of a parabola is $(1,5)$ and the focus is $(4,5)$. What is an equation of the directrix?

13. The directrix of a parabola is the line with equation $x = 2$, and the endpoints of the latus rectum are $(6,6)$ and $(6,-2)$. Write an equation of this parabola.

14. Write an equation of the circle having the segment with endpoints $(-1,3)$ and $(3,-3)$ as a diameter.

15. What is an equation of the conic with eccentricity of $\frac{3}{2}$ and foci at $(3,8)$ and $(3,2)$?

16. (a) Write an equation of the family of hyperbolas with center at $(2,-3)$ and asymptotes with slopes $\frac{2}{5}$ and $-\frac{2}{5}$.

(b) Find an equation of the member of that family which contains the point $(22,7)$.

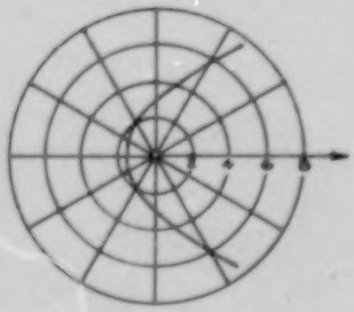
Answers for Illustrative Test Items

1. (In some routine cases the graphs are omitted.)

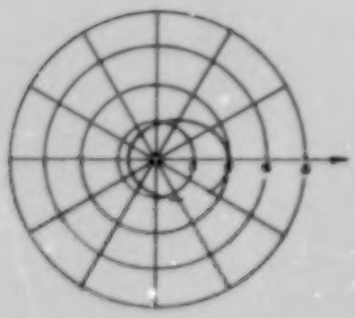
(a) Circle, center $(0,0)$, radius 5.

(b) Circle, center $(1, \frac{\pi}{2})$, radius 1.

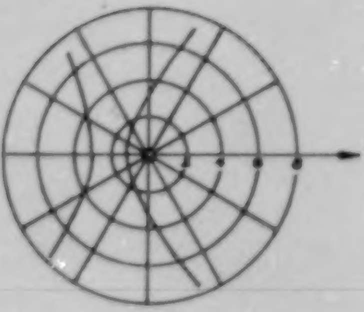
(c) Parabola $r = \frac{3}{1 - \cos\theta}$



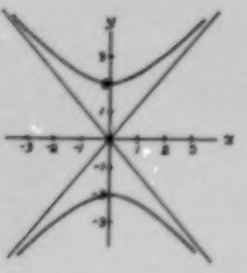
(d) Ellipse $r = \frac{2}{1 - .5 \cos \theta}$



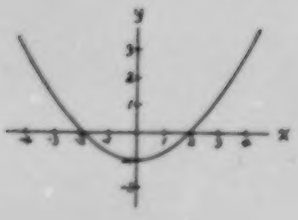
(e) Hyperbola $r = \frac{1.5}{1 - 2 \cos \theta}$



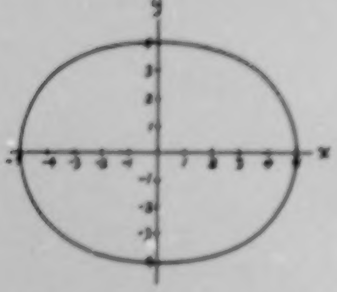
(f) Hyperbola $-\frac{x^2}{3} + \frac{y^2}{4} = 1$



(g) Parabola $x^2 = 4(y + 1)$



(h) Ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$



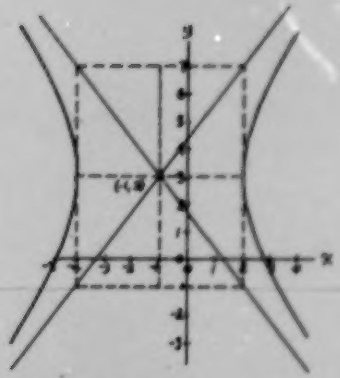
(i) Point-ellipse (2,-4)

$$9(x - 2)^2 + 4(y + 4)^2 = 0$$

(j) $(x + 1)^2 - 25(y - 2)^2 = 0$ or $(x + 5y - 9)(x - 5y + 11) = 0$

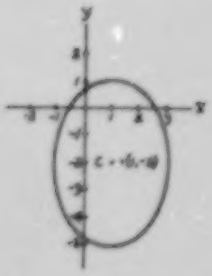
Two lines, equations $x + 5y - 9 = 0$ and $x - 5y + 11 = 0$.

(k) Hyperbola $\frac{(x + 1)^2}{9} - \frac{(y - 3)^2}{16} = 1$



(f) $\frac{(x - 1)^2}{4} + \frac{(y + 2)^2}{9} = 1$

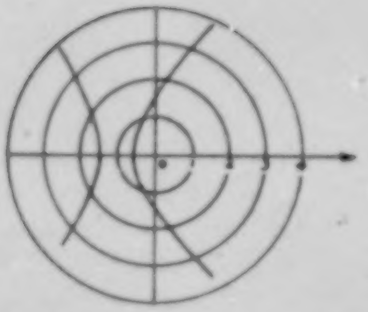
Ellipse



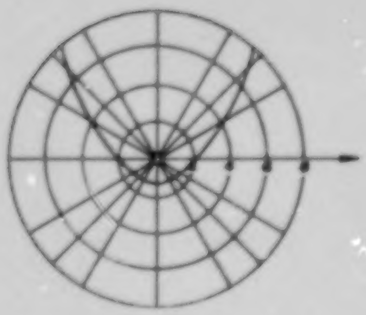
2. (a) Circle; $x^2 + y^2 = \frac{49}{4}$

(b) Hyperbola;

$$\frac{(x + 2)^2}{1} - \frac{y^2}{3} = 1$$

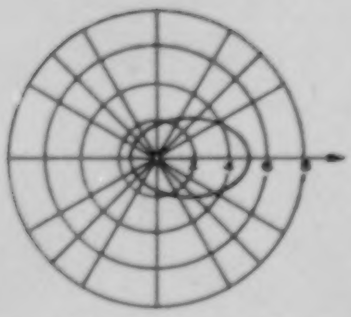


(c) Parabola $x^2 = 4(y + 1)$



(d) Ellipse

$$\frac{(x - 2)^2}{9} + \frac{y^2}{5} = 1$$



3. (a) Ellipse; $e = \frac{1}{3}$.

(b) Hyperbola; $e = \frac{3}{2}$.

(c) Circle; no eccentricity (or $e = 0$).

(d) Parabola; $e = 1$

4. $V = (2,-2)$

5. $a = 2$, $b = 2\sqrt{3}$

6. $y - 4 = \frac{3}{4}(x + 3)$, or $3x - 4y + 25 = 0$

7. $3x - 2y - 9 = 0$

8. (a) Parabola; line symmetry, $\theta = 0$

   (b) Circle; point symmetry, $(\frac{\sqrt{2}}{2}, \frac{\pi}{4})$; line symmetry, every line through the center.

   (c) Hyperbola; point symmetry, $(0,0)$; line symmetry, $x = 0$ and $y = 0$.

   (d) Ellipse; point symmetry, $(2,-4)$; line symmetry, $x = 2$, $y = -4$.

   (e) Hyperbola; point symmetry, $(-3,1)$; line symmetry, $x = -3$ and $y = 1$.

   (f) Parabola; line symmetry, $x = 3$.

9. $$(x - 3)^2 + (y + 1)^2 = \left(\frac{|2(3) + 5(-1) - 5|}{\sqrt{4 + 25}}\right)^2 = \frac{16}{29}$$

   or $29x^2 + 29y^2 - 174x + 58y + 274 = 0$

10. $e = \frac{4}{5}$

11. $2ae = 2\sqrt{13}$

12. $x = -2$

13. $(y - 2)^2 = 8(x - 4)$

14. $C = (1,0)$; $r = \sqrt{13}$

    $(x - 1)^2 + y^2 = 13$

15. Hyperbola; $ae = 3$, $a = 2$, $b = \sqrt{5}$; $c = (3,5)$

    $$-\frac{(x - 3)^2}{5} + \frac{(y - 5)^2}{4} = 1$$

16. (a) $\frac{(x - 2)^2}{25} - \frac{(y + 3)^2}{4} = k$

    (b) $\frac{(x - 2)^2}{25} - \frac{(y + 3)^2}{4} = -9$

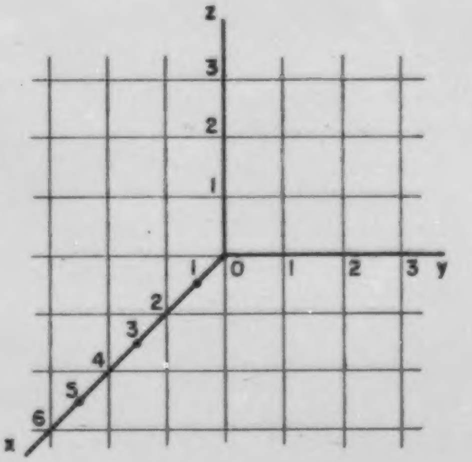
Teachers' Commentary

Chapter 8

THE LINE AND THE PLANE IN 3-SPACE

Parts of this chapter will be familiar to some classes. Time saved when this is the case may permit study of some of the supplementary chapters.

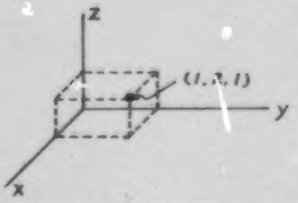
Many teachers have a favorite method of teaching students to make sketches of solids. If you do not have such a preferred method, you might like to try this. Have the students use squared paper; tell them to draw $\overleftrightarrow{OY}$ horizontal, $\overleftrightarrow{OZ}$ vertical, and $\overleftrightarrow{OX}$ at an angle of 45° with the negative end of the y-axis. Choose a suitable length for the unit on the y- and z-axis, and on the x-axis let the diagonal of the unit square measure two units. This is a convenient way of getting units, and makes a rather satisfactory drawing.



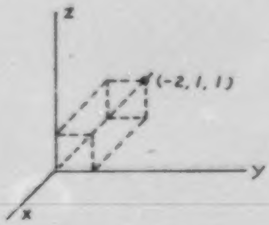
The formulas for point of division in Section 8-2 apply for both internal and external points of division. In Exercise 9, parts (b) to (f), of Exercises 8-2, the distances are considered to be directed, and two points are found.

Exercises 8-2

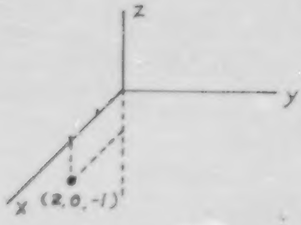
1. (a)



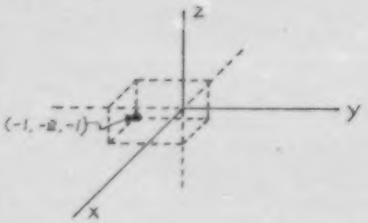
(b)



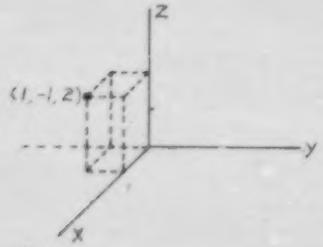
(c)



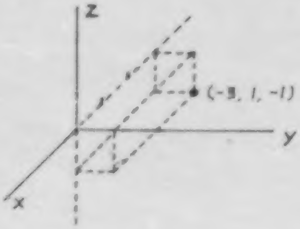
(f)



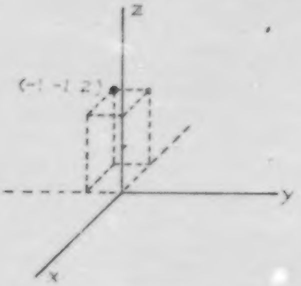
(d)



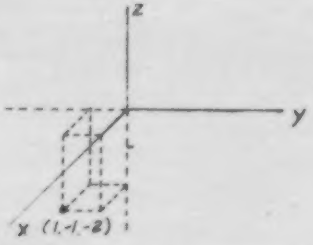
(g)



(e)



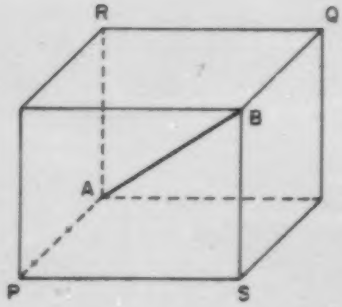
(h)



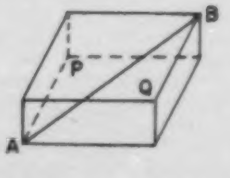
2. $d(O,P) = \sqrt{14}$, $d(O,Q) = \sqrt{14}$, $d(P,R) = \sqrt{30}$, $d(Q,R) = 5\sqrt{2}$.

3. Midpoint of $\overline{OP} = (\frac{1}{2}, 1, \frac{3}{2})$, midpoint of $\overline{PR} = (\frac{3}{2}, -\frac{1}{2}, 2)$.

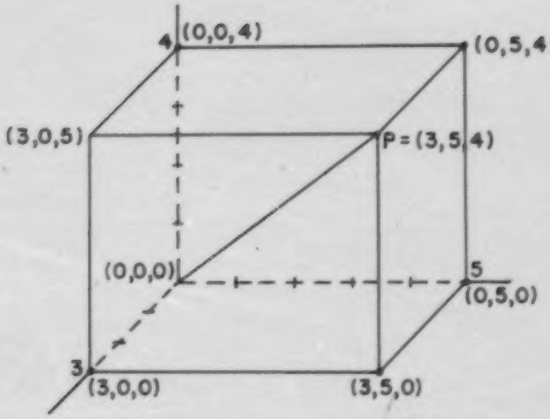
4. (a)



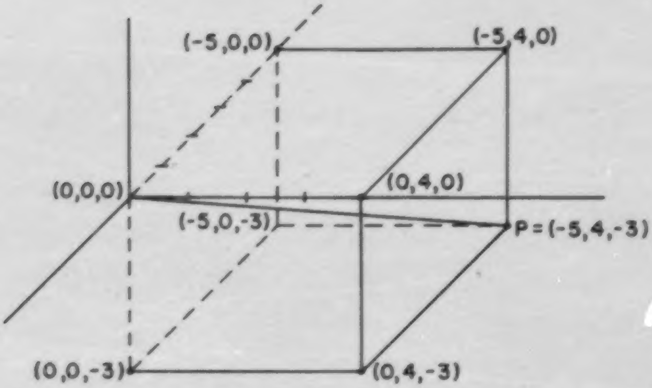
(b)



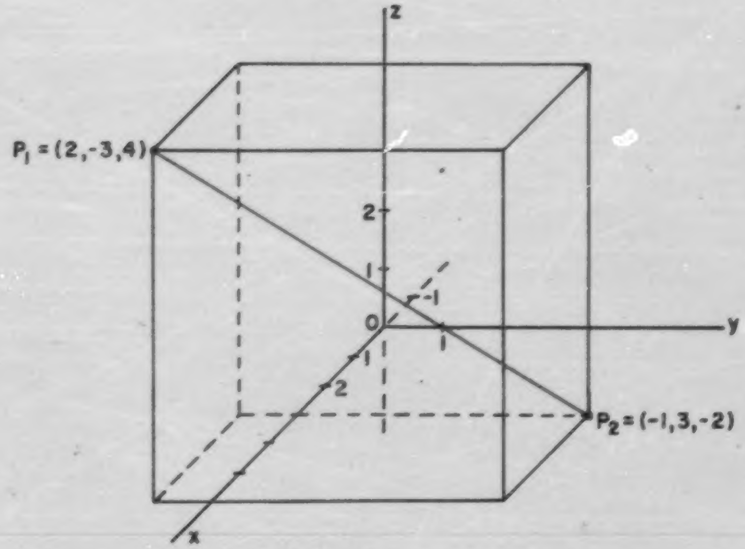
5.


4
(0,0,4)
(0,5,4)
(3,0,5)
P = (3,5,4)
(0,0,0)
5
(0,5,0)
3
(3,0,0)
(3,5,0)

6.


(-5,0,0)
(-5,4,0)
(0,0,0)
(0,4,0)
(-5,0,-3)
P = (-5,4,-3)
(0,0,-3)
(0,4,-3)

7. (a)


z
P₁ = (2,-3,4)
2
1
-1
0
y
1
1
2
P₂ = (-1,3,-2)
x

(b) Projections	P_1	P_2
on x-axis	(2,0,0)	(-1,0,0)
on y-axis	(0,-3,0)	(0,3,0)
on z-axis	(0,0,4)	(0,0,-2)
on xy-plane	(2,-3,0)	(-1,3,0)
on yz-plane	(0,-3,4)	(0,3,-2)
on xz-plane	(2,0,4)	(-1,0,-2)

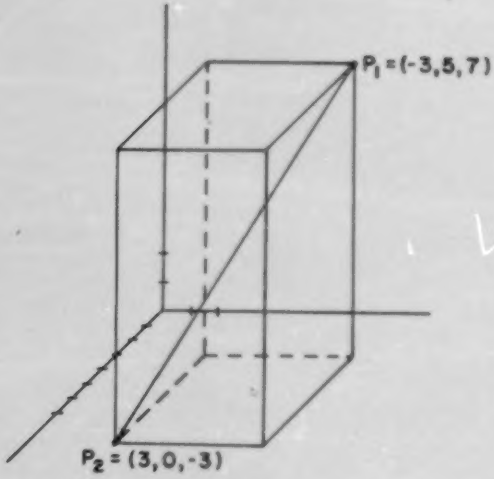
(c) $d(P_1,P_2) = 9$

lengths of projections

on x-,y-, and z-axes: 3,6,6

on xy-, yz-, and xz-planes: $3\sqrt{5}$, $6\sqrt{2}$, $3\sqrt{5}$

8. (a)



(b) Projections	P_1	P_2
on x-axis	(-3,0,0)	(3,0,0)
on y-axis	(0,5,0)	(0,0,0)
on z-axis	(0,0,7)	(0,0,-3)
on xy-plane	(-3,5,0)	(3,0,0)
on yz-plane	(0,5,7)	(0,0,-3)
on xz-plane	(-3,0,7)	(3,0,-3)

(c) $d(P_1, P_2) = \sqrt{161}$

lengths of projections

on x- , y- , and z-axes 6, 5, 10

on xy- , yz- , and xz-planes $\sqrt{61}$, $5\sqrt{5}$, $2\sqrt{34}$

9. (a) $(\frac{1}{2}, -\frac{1}{2}, 2)$

(b) $(\frac{4}{3}, -\frac{5}{3}, \frac{10}{3})$ or $(8, -11, 14)$

(c) $(\frac{9}{8}, -\frac{11}{8}, 3)$ or $(\frac{21}{2}, -\frac{29}{2}, 18)$

(d) $(-\frac{1}{8}, \frac{3}{8}, 1)$ or $(-\frac{19}{2}, \frac{27}{2}, -14)$

(e) $(0, \frac{1}{5}, \frac{6}{5})$ or $(6, -\frac{41}{5}, \frac{54}{5})$

(f) $(\frac{34}{3}, -\frac{47}{3}, \frac{58}{3})$ or $(-\frac{16}{3}, \frac{23}{3}, -\frac{22}{3})$

10. $d(A,B) = \sqrt{14}$
$d(A,C) = \sqrt{34}$
$d(B,C) = \sqrt{20}$
Right triangle.

Challenge Problem

The question of how we know there are three mutually perpendicular lines through a point in space is intended as a warning to the students against the uncritical use of intuition. It is not a trivial question. In terms of the development in the SMSG Geometry it can be answered as follows. By a postulate, there are at least four points in space, so we can select O and another point, P. By another postulate there is a unique line L_1, containing O and P. By a theorem, there exists a unique plane α through O perpendicular to $\overleftrightarrow{OP}$. By a postulate there is another point Q in α. By a postulate there is a unique line L_2 through O and Q, and it is perpendicular to $\overleftrightarrow{OP}$. Finally, by a theorem there is a unique line in α through O perpendicular to L_2, and by another theorem it is perpendicular to L_1 too.

The argument by which the parametric representation is obtained is rather tricky and should probably be gone over in class very carefully. It may help to show that

$$z = z_0 + s(z_1 - z_0)$$

for suitable s by noting that $z_1 - z_0 \neq 0$ and hence $s = \frac{z - z_0}{z_1 - z_0}$ will do. In the final step the argument is that from the parametric equations for L' and L" we see that

$$y = y_0 + s(y_1 - y_0)$$

for suitable t and that

$$y = y_0 + s(y_1 - y_0)$$

for suitable s. Since $y_1 - y_0 \neq 0$,

$$s = \frac{y - y_0}{y_1 - y_0} = t.$$

Students are often intrigued by the idea of a 4-dimensional space, so they may enjoy our brief discussion of the notion. If it is taken up, you should try to make it clear that we are not introducing a coordinate system into a space which is given (by a system of postulates) but instead are defining a "space" which is in many ways like the space of ordinary geometry.

In 3-space, as in 2-space, a line has not just one, but many representations. Only one is given in this commentary, except where the directions specifically ask for two. A student should be allowed to write any correct representation, but should be able to show that his representation is equivalent to any desired representation.

In Exercise 2 of Exercises 8-3, you may want to have the students consider further the cases in which symmetric representation is not possible. In part (d), for example, since one of the direction numbers is zero, symmetric equations cannot be written. However, t can be eliminated and the equations of two planes containing the line written: $z - 1 = 0$, $x + 5y + 13 = 0$. In part (a), with two direction numbers equal to zero, we have at once the equations of two such planes: $y = 2$, $z = 3$.

Exercises 8-3

1. (a) $\begin{cases} x = 1 + t(1) \\ y = 2 + t(0) \\ z = 3 + t(0) \end{cases}$

   (b) $\begin{cases} x = -3 + t(0) \\ y = -2 + t(0) \\ z = 1 + t(1) \end{cases}$

   (c) $\begin{cases} x = 1 + t(-4) \\ y = 2 + t(-4) \\ z = 3 + t(-2) \end{cases}$

   (d) $\begin{cases} x = -3 + t(5) \\ y = -2 + t(-1) \\ z = 1 + t(0) \end{cases}$

   (e) $\begin{cases} x = 0 + t(-4) \\ y = 0 + t(-4) \\ z = 0 + t(-2) \end{cases}$

   (f) $\begin{cases} x = 0 + t(5) \\ y = 0 + t(-1) \\ z = 0 + t(0) \end{cases}$

   (g) $\begin{cases} x = 0 + t(1) \\ y = 0 + t(2) \\ z = 0 + t(3) \end{cases}$

   (h) $\begin{cases} x = 1 + t(-1) \\ y = 2 + t(-2) \\ z = 3 + t(0) \end{cases}$

   (i) $\begin{cases} x = 1 + t(5) \\ y = 2 + t(-1) \\ z = 3 + t(0) \end{cases}$

   (j) $\begin{cases} x = 2 + t(-4) \\ y = -3 + t(-4) \\ z = 1 + t(-2) \end{cases}$

2. The symmetric form exists in parts (c), (e), (g) and (j)

   (c) $\frac{x - 1}{-4} = \frac{y - 2}{-4} = \frac{z - 3}{-2}$

   (e) $\frac{x}{-4} = \frac{y}{-4} = \frac{z}{-2}$

   (g) $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$

   (j) $\frac{x - 2}{-4} = \frac{y + 3}{-4} = \frac{z - 1}{-2}$

3. (a) $(1,0,0)$

   (b) $(0,0,1)$

   (c) $(-\frac{2}{3}, -\frac{2}{3}, -\frac{1}{3})$

   (d) $(\frac{5}{\sqrt{26}}, -\frac{1}{\sqrt{26}}, 0)$

   (e) $(-\frac{2}{3}, -\frac{2}{3}, -\frac{1}{3})$

   (f) $(\frac{5}{\sqrt{26}}, -\frac{1}{\sqrt{26}}, 0)$

   (g) $(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}})$

(h) $\left(-\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}, 0\right)$

(i) $\left(\frac{5}{\sqrt{26}}, -\frac{1}{\sqrt{26}}, 0\right)$

(j) $\left(-\frac{2}{3}, -\frac{2}{3}, -\frac{1}{3}\right)$

4. (a) $\begin{cases} x = 1 + t(-1) \\ y = 1 + t(-2) \\ z = -2 + t(1) \end{cases}$ $\begin{cases} x = 0 + t(1) \\ y = -1 + t(2) \\ z = -1 + t(-1) \end{cases}$ $(-1, -3, 0)$

(b) $\begin{cases} x = -1 + t(-1) \\ y = -1 + t(0) \\ z = -1 + t(2) \end{cases}$ $\begin{cases} x = -2 + t(1) \\ y = -1 + t(0) \\ z = 1 + t(-2) \end{cases}$ $(-3, -1, 3)$

(c) $\begin{cases} x = 4 + t(-3) \\ y = 2 + t(-4) \\ z = 1 + t(3) \end{cases}$ $\begin{cases} x = 1 + t(3) \\ y = -2 + t(4) \\ z = 4 + t(-3) \end{cases}$ $(-2, -6, 7)$

(d) $\begin{cases} x = -3 + t(4) \\ y = 1 + t(1) \\ z = 1 + t(-2) \end{cases}$ $\begin{cases} x = 1 + t(-4) \\ y = 2 + t(-1) \\ z = -1 + t(2) \end{cases}$ $(5, 3, -3)$

5. (a) $\left(\frac{-1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$; $114^\circ, 145^\circ, 66^\circ$; $\left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}\right)$: $66^\circ, 35^\circ, 114^\circ$.

(b) $\left(\frac{-1}{\sqrt{5}}, \frac{0}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$: $117^\circ, 90^\circ, 27^\circ$. $\left(\frac{1}{\sqrt{5}}, \frac{0}{\sqrt{5}}, \frac{-2}{\sqrt{5}}\right)$: $63^\circ, 90^\circ, 153^\circ$.

(c) $\left(\frac{-3}{\sqrt{34}}, \frac{-4}{\sqrt{34}}, \frac{3}{\sqrt{34}}\right)$; $121^\circ, 133^\circ, 59^\circ$ $\left(\frac{3}{\sqrt{34}}, \frac{4}{\sqrt{34}}, \frac{-3}{\sqrt{34}}\right)$; $59^\circ, 47^\circ, 121^\circ$.

(d) $\left(\frac{4}{\sqrt{21}}, \frac{1}{\sqrt{21}}, \frac{-2}{\sqrt{21}}\right)$; $29^\circ, 77^\circ, 116^\circ$. $\left(\frac{-4}{\sqrt{21}}, \frac{-1}{\sqrt{21}}, \frac{2}{\sqrt{21}}\right)$; $151^\circ, 103^\circ, 64^\circ$

6. **x-axis** (1,0,0)
y-axis (0,1,0)
z-axis (0,0,1)

7. $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

8. (a) No
   (b) Yes
   (c) Yes
   (d) Yes

9. Lines with equations (a) and (f) are parallel; so are (b) and (d).

10. $L_1 : \frac{x - 2}{3} = \frac{y - 1}{-2} = \frac{z + 1}{-1}$ $\qquad L_2 : \frac{x + 1}{1} = \frac{y - 2}{2} = \frac{z - 4}{-1}$

    $L_3 : \frac{x - 3}{2} = \frac{y + 5}{-3} = \frac{z}{4}$

    L_4 cannot be written in symmetric form.

11.
$$d(P_1,P_2) = \sqrt{[(x_0 + \ell t_1) - (x_0 + \ell t_2)]^2 + [(y_0 + mt_1) - (y_0 + mt_2)]^2 + [(z_0 + nt_1) - (z_0 + nt_2)]^2}$$
$$= \sqrt{(\ell t_1 - \ell t_2)^2 + (mt_1 - mt_2)^2 + (nt_1 - nt_2)^2}$$
$$= \sqrt{(\ell^2 + m^2 + n^2)(t_1 - t_2)^2}$$
$$= \sqrt{\ell^2 + m^2 + n^2}\,|t_1 - t_2|$$

The distance between any two points on a line with the given parametirc representation is a constant multiple $(\sqrt{\ell^2 + m^2 + n^2})$ of the absolute value of the difference of the values of the parameter that give the points. If the direction numbers are normalized, the distance is equal to the absolute value of the difference of the parameters.

12. Suppose L is in or parallel to the xy-plane. In that plane, L would have the parametric representation

$$x = x_0 + t(x_1 - x_0)$$
$$y = y_0 + t(y_1 - y_0) .$$

The z-coordinate of every point on L would be the same number, so $z = z_0$. Thus Equations (3) would represent the line L ; similarly, they would represent a line parallel to either of the other coordinate planes.

Challenge Problems

1. For all values of t the x-coordinate of the point on L will be 2 so the line is in the plane $x = 2$. Similarly for every point $P(x,y,z)$ in the plane $3y - z = -5$ there is a value of t such that $y = -1 + t$ and $z = 2 + 3t$.

   So $L : x = 2 , y = -1 + t , z = 2 + 3t$ lies in the intersection of the planes $x = 2$ and $3y - z = -5$.

2. $x = 2 \quad z = -1$

3. $$\begin{cases} x = x_0 + t(x_1 - x_0) \\ y = y_0 + t(y_1 - y_0) \\ z = z_0 + t(z_1 - z_0) \\ w = w_0 + t(w_1 - w_0) \end{cases}$$

   If $P_2(x_2, y_2, z_2, w_2)$ is on L then there is a number

   $$t_2 \neq 0 \text{ such that } \begin{cases} x_2 = x_0 + t_2(x_1 - x_0) \\ y_2 = y_0 + t_2(y_1 - y_0) \\ z_2 = z_0 + t_2(z_1 - z_0) \\ w_2 = w_0 + t_2(w_1 - w_0) \end{cases}$$

   But then

   $$\begin{cases} x_1 = x_0 + (\frac{1}{t_2})(x_2 - x_0) \\ y_1 = y_0 + (\frac{1}{t_2})(y_2 - y_0) \\ z_1 = z_0 + (\frac{1}{t_2})(z_2 - z_0) \\ w_1 = w_0 + (\frac{1}{t_2})(w_2 - w_0) \end{cases}$$

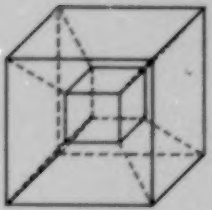
   So P_1 is on the line through P_0 and P_2.

4. On the coordinate axes, $(x_0,0,0,0)$, $(0,y_0,0,0)$, $(0,0,z_0,0)$, $(0,0,0,w_0)$.

   On the coordinate planes, $(x_0,y_0,0,0)$, $(0,y_0,z_0,0)$, $(0,0,z_0,w_0)$, $(x_0,0,0,w_0)$, $(x_0,0,z_0,0)$, $(0,y_0,0,w_0)$.

   On the coordinate hyperplanes, $(x_0,y_0,z_0,0)$, $(0,y_0,z_0,w_0)$, $(x_0,0,y_0,z_0)$, $(x_0,y_0,0,w_0)$.

5.



In 3-space $V - E + F = 2$ where V is the number of vertices, E, the number of edges, F, the number of faces.

In 4-space the polyhedron is made up of vertices (0-dimensional), edges (1-dimensional), faces (2-dimensional), and hyperfaces (3-dimensional). In the picture the hyperfaces are represented as truncated pyramids with bases on faces of the figures that appear as inner and outer cubes.

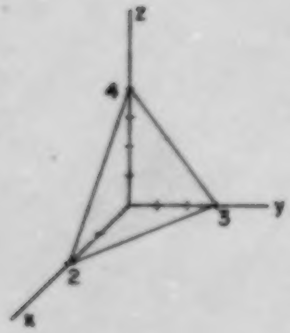
$V - E + F - H = 0$

Exercises 8-4

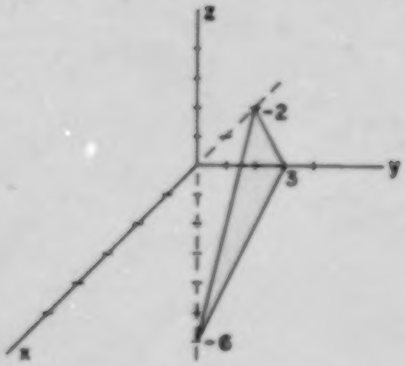
1. $2x - y - 4z + 6 = 0$
2. $x + 4y - 5z - 6 = 0$
3.

	Intercepts			Traces in		
	x-	y-	z-	xy-plane	yz-plane	xz-plane
(a)	2	3	4	$3x + 2y - 6 = 0$	$4y + 3z - 12 = 0$	$2x + z - 4 = 0$
(b)	5	2	10	$2x + 5y - 10 = 0$	$5y + z - 10 = 0$	$2x + z - 10 = 0$
(c)	$2\frac{1}{2}$	-5	-2	$2x - y - 5 = 0$	$2y + 5z + 10 = 0$	$4x - 5z - 10 = 0$
(d)	-2	3	-6	$3x - 2y + 6 = 0$	$-2y + z + 6 = 0$	$3x + z + 6 = 0$
(e)	4	-3	none	$3x - 4y - 12 = 0$	$y + 3 = 0$	$x - 4 = 0$
(f)	none	-4	$\frac{5}{2}$	$y + 4 = 0$	$5y - 8z + 20 = 0$	$-2z + 5 = 0$
(g)	0	0	0	$x - 2y = 0$	$-3y + z = 0$	$3x + 2z = 0$
(h)	none	0	0	$y = 0$	$3y - 5z = 0$	$z = 0$
(i)	7	none	none	$x - 7 = 0$	none	$x - 7 = 0$
(j)	none	none	$-\frac{9}{2}$	none	$2z + 9 = 0$	$2z + 9 = 0$

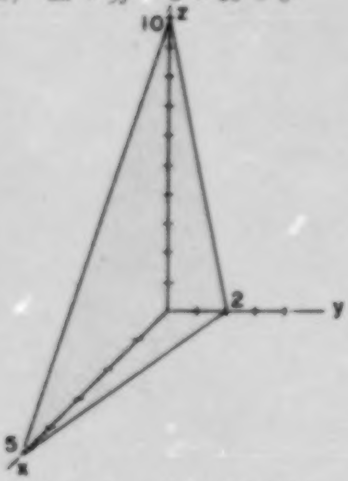
(a) $6x + 4y + 3z - 12 = 0$



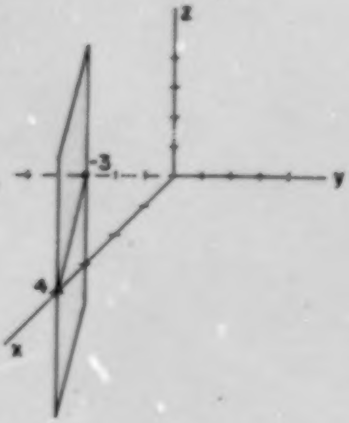
(d) $3x - 2y + z + 6 = 0$



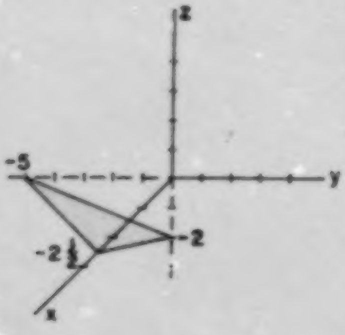
(b) $2x + 5y + z - 10 = 0$



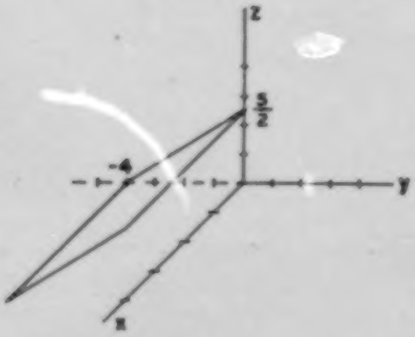
(e) $3x - 4y - 12 = 0$



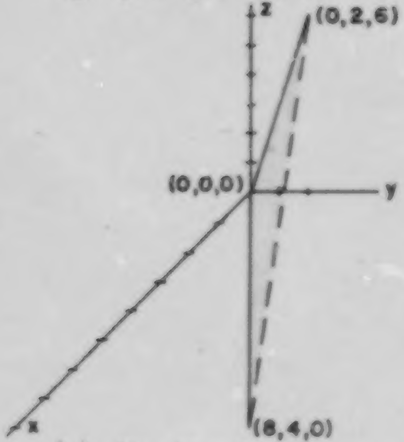
(c) $4x - 2y - 5z - 10 = 0$


z
-5
y
-2
$-2\frac{1}{2}$
x

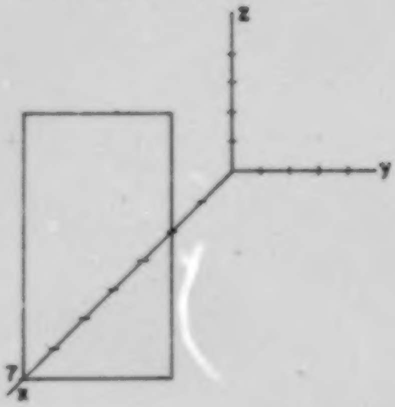
(f) $5y - 8z + 20 = 0$


z
$\frac{5}{2}$
-4
y
x

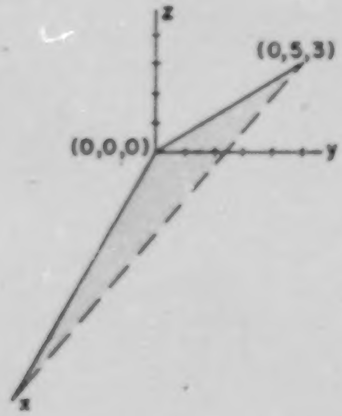
(g) 3x - 6y + 2z = 0


z
(0,2,6)
(0,0,0)
y
x
(8,4,0)

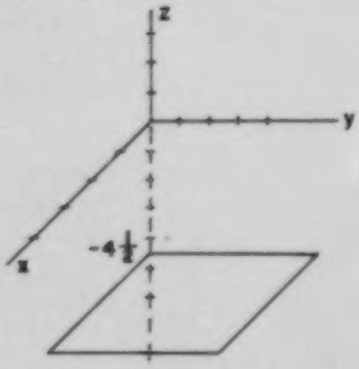
(i) x - 7 = 0



(h) 3y - 5z = 0

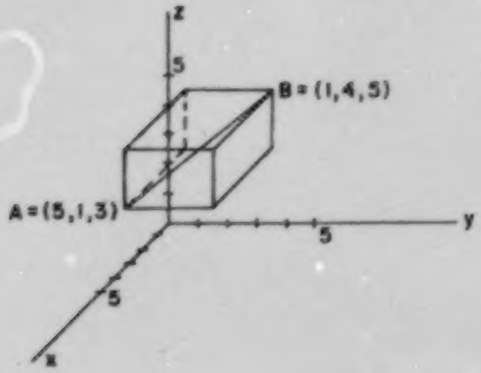


(j) 2z + 9 = 0

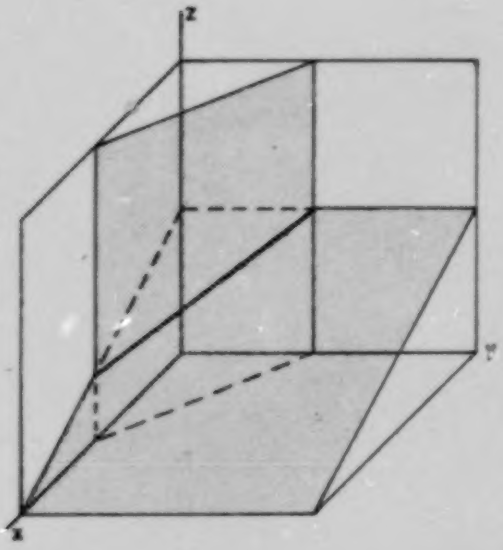

z
y
x
$-4\frac{1}{2}$

4. (a) ax + by + cz = 0
   (b) cz + d = 0
   (c) ax + d = 0
   (d) ax + by + d = 0
   (e) by + cz + d = 0
   (f) ax + cz + d = 0

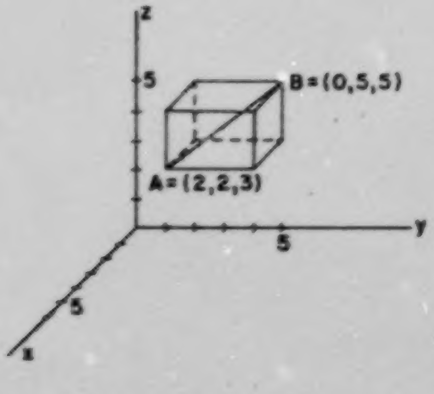
5. (a)


z
5
B = (1,4,5)
A = (5,1,3)
y
5
5
x

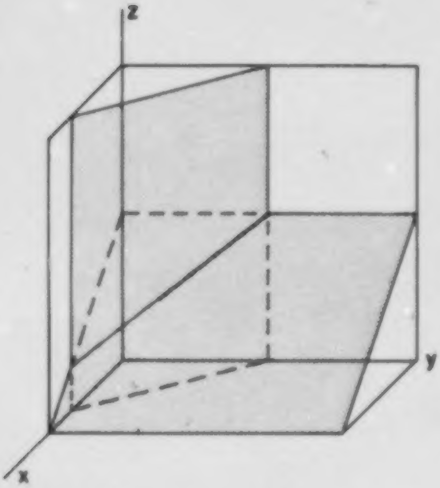
(b)



6. (a)


z
5
B = (0,5,5)
A = (2,2,3)
y
5
5
x

(b)



7. $(\ell, m, n) = (3,-2,5)$; $(\lambda, \mu, \nu) = (\frac{3}{\sqrt{38}}, \frac{-2}{\sqrt{38}}, \frac{5}{\sqrt{38}})$

8. $(\ell, m, n) = (4,-1,0)$; $(\lambda, \mu, \nu) = (\frac{4}{\sqrt{17}}, \frac{-1}{\sqrt{17}}, 0)$

9. (a) $\frac{4}{\sqrt{61}}$
   (b) 0
   (c) $\frac{28}{\sqrt{45}}$
   (d) $\frac{1}{\sqrt{14}}$
   (e) $\frac{23}{5}$
   (f) $\frac{14}{\sqrt{89}}$
   (g) $\frac{11}{7}$
   (h) $\frac{4}{\sqrt{34}}$
   (i) 8
   (j) $\frac{13}{2}$

10. (a) $\frac{7}{\sqrt{61}}$
   (b) $\frac{11}{\sqrt{30}}$
   (c) $\frac{3}{\sqrt{5}}$
   (d) 0
   (e) 5
   (f) $\frac{49}{\sqrt{89}}$
   (g) $\frac{23}{7}$
   (h) $\frac{17}{\sqrt{34}}$
   (i) 6
   (j) $\frac{7}{2}$

11. (a) $-8x + 3y - 7z + 23 = 0$
    (b) $2x - 3z - 1 = 0$

12. (a) $3x - 2y + z + 4 = 0$
    (b) $x - 2z + 5 = 0$

13. Let the equation of the plane be $Ax + By + Cz + D = 0$.
    Since the plane contains $(a,0,0)$, $(0,b,0)$, and $(0,0,c)$, we have

$$\begin{aligned} & Aa + D = 0 && \text{or} && A = -\frac{D}{a} , \\ & Bb + D = 0 && \text{or} && B = -\frac{D}{b} , \\ \text{and}\quad & Cc + D = 0 && \text{or} && C = -\frac{D}{c} . \end{aligned}$$

Thus if $D = -1$, an equation is $\frac{x}{a} + \frac{y}{b} + \frac{c}{z} = 1$.

14. (a) $\frac{x}{1} + \frac{y}{3} + \frac{z}{4} = 1$

    (b) $\frac{x}{-2} + \frac{y}{5} + \frac{z}{-3} = 1$

15. (a) $x - 12y + 3z - 7 = 0$
    (b) $3x + 8y + 2z - 15 = 0$

16. One method would be to find an equation of the plane determined by points A , B , C $(2x - y + z - 1 = 0)$ and then check that point D is a point of this plane.

17. (a) $x + y + 2z - 2 = 0$
    (b) $3y - 2z + 1 = 0$

18. The proof given in Intermediate Mathematics may be familiar; it follows.

Let $P = (x,y,z)$ be any point on the plane that is the set of points equidistant from $O = (0,0,0)$ and $Q = (ka,kb,kc)$ where

$$k = \frac{-2d}{a^2 + b^2 + c^2}$$

Then, since $d(P,O) = d(P,Q)$,

$$x^2 + y^2 + z^2 = (x - ka)^2 + (y - kb)^2 + (z - kc)^2 ,$$

or
$$0 = -2kax + k^2a^2 - 2kby + k^2b^2 - 2kc + k^2c^2 ,$$

which becomes $2k(ax + by + cz) = k^2(a^2 + b^2 + c^2)$

or
$$ax + by + cz = \frac{k}{2}(a^2 + b^2 + c^2) .$$

By substituting the value of k, this equation becomes

$$ax + by + cz + d = 0 .$$

This argument is reversible. This means that any point P whose coordinates satisfy $ax + by + cz + d = 0$ is equidistant from the points O and Q. Hence $ax + by + cz + d = 0$ is the equation of a plane.

Note. If $d = 0$, it follows that $k = 0$; the two points coincide, and no plane is determined. In this case we use the symmetric points (a,b,c) and $(-a,-b,-c)$ and carry through the same steps as above.

8-5 The definition of a vector as a set of equivalent directed segments makes the extension to 3-space almost trivial. Since any member of the set may represent the vector, we are free to choose those representatives which most simplify our models or diagrams. In Chapter 3 we stressed the freedom, but we did not attempt initially to pursue all the consequences of this freedom. At this point it may be helpful to review the earlier material briefly and to point out that all vectors which have representatives on parallel lines also have representatives on the same line. Once this property of vectors is understood, the approach to 3-space should follow more easily.

The proof of Example 1 assumes no more knowledge of prisms than the material presented in the SMSG Geometry. If students have had additional training in solid geometry, they should be able to develop a more concise proof.

In Chapter 3 we approached vectors from a purely geometric point of view before introducing any analysis using components. Here we have adopted the same approach, but in almost any application of vectors it is more convenient to use representations in component form.

For simplicity we use i, j, and k to represent basis vectors without the usual symbols indicating vector quantities. Consequently it may be necessary to stress that these are indeed vectors.

Exercises 8-5

1. (a) 0
(b) 0
(c) 0
(d) 1
(e) 1
(f) 1
(g) 0
(h) 7

2. (a) 0
(b) 0
(c) 0
(d) 1
(e) 1
(f) 1
(g) 0
(h) $\frac{\sqrt{7}}{2\sqrt{3}}$ or $\frac{\sqrt{21}}{6}$

3. $r = \pm \frac{1}{3}$

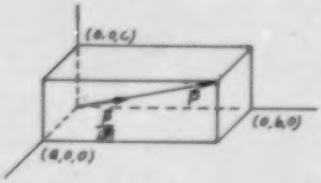
4. (a) [14,-3,3]
(b) [-7,16,-9]
(c) [-2,17,-9]
(d) [6,0,2]
(e) [14,10,-2]
(f) [-18,-4,-6]

5. (a) [6,-2,2]
(b) $[\frac{17}{5}, -\frac{12}{5}, \frac{9}{5}]$
(c) $[-\frac{1}{3}, -\frac{1}{3}, -\frac{2}{3}]$
(d) $[0, -\frac{2}{3}, 0]$
(e) [-7,0,1]
(f) $[-\frac{13}{6}, 0, -\frac{1}{6}]$

6. (a) -1
(b) -6
(c) 24
(d) -50
(e) 0
(f) -70
(g) 257
(h) 4
(i) -231
(j) 42

7. $\vec{A} \cdot \vec{A}$ is a real number defined $|\vec{A}||\vec{A}| \cos \theta$ where θ is the angle between $\vec{A}$ and $\vec{A}$; i.e., $\theta = 0^\circ$. So $\vec{A} \cdot \vec{A} = |\vec{A}| \cdot |\vec{A}| = |\vec{A}|^2$. $|\vec{A}|^2$ and $|\vec{A}|^3$ are real numbers. $\vec{A} \cdot \vec{A} \cdot \vec{A}$ is not defined unless a convention about the order of mutliplication is made, but in any case, e.g., $(\vec{A} \cdot \vec{A}) \cdot \vec{A} = |\vec{A}|^2 \vec{A}$, the product is a vector, not a number.

8. $\frac{P}{|P|} = \frac{a}{\sqrt{a^2 + b^2 + c^2}}\,i + \frac{b}{\sqrt{a^2 + b^2 + c^2}}\,j + \frac{c}{\sqrt{a^2 + b^2 + c^2}}\,k$.



A vector of magnitude one in the direction of P .

9. $k = -\frac{9}{2}$

10. $\frac{1}{\sqrt{3}}$

11. The line segment joining the endpoint of $\vec{A}$ and $\vec{B}$ is parallel to $\vec{A} - \vec{B}$. But $\vec{A} - \vec{B} = i + 4j$ which lies in the xy-plane. Hence $\overleftrightarrow{AB}$ is parallel to the xy-plane.

12. If $\vec{c} \perp \vec{a}$ and $\vec{c} \perp \vec{b}$, we must show that $\vec{c} \perp (\vec{a} + \vec{b})$ or that $\vec{c} \cdot (\vec{a} + \vec{b}) = 0$. But $\vec{c} \cdot (\vec{a} + \vec{b}) = \vec{c} \cdot \vec{a} + \vec{c} \cdot \vec{b}$. Since $\vec{c} \perp \vec{a}$, $\vec{c} \cdot \vec{a} = 0$, and since $\vec{c} \perp \vec{b}$, $\vec{c} \cdot \vec{b} = 0$. Therefore, $\vec{c} \cdot (\vec{a} + \vec{b}) = 0$ and $\vec{c} \perp (\vec{a} + \vec{b})$.

13. $[0,0,\pm 1]$ or $\pm k$.

14. $-7i + 6j - k$

$7ci - 6cj + ck$, where $c \neq 0$.

15. $\angle A = 124^{\circ}$

$\angle B = 32^{\circ}$

$\angle C = 24^{\circ}$

16. $\vec{P} + r[0,1,-\frac{b}{c}]$ is a parametric representation of a line through $\vec{P} = [a,b,c] \neq [0,0,0]$ which is perpendicular to $\vec{P}$; i.e., there is such a line for each r .

17. We wish to prove that $r\vec{A} = [ra_1, ra_2, ra_3]$.

Since $\vec{A} = [a_1, a_2, a_3]$,

$r\vec{A} = r[a_1, a_2, a_3]$.

Does $r[a_1, a_2, a_3] = [ra_1, ra_2, ra_3]$?

If two vectors are equal, their magnitudes and directions are equal.

magnitude of $r[a_1,a_2,a_3] = |r|\sqrt{a_1^2 + a_2^2 + a_3^2}$

magnitude of $[ra_1,ra_2,ra_3] = \sqrt{r^2a_1^2 + r^2a_2^2 + r^2a_3^2}$

$$|r|\sqrt{a_1^2 + a_2^2 + a_3^2} .$$

If r is positive or negative the directions of the two are equal. Therefore the vectors are equal.

18. (a) We wish to prove that

$$\vec{X} \cdot (\vec{Y} + \vec{Z}) = \vec{X} \cdot \vec{Y} + \vec{X} \cdot \vec{Z} ;$$

We expand the left-hand member to obtain

$$\begin{aligned}
&[x_1,x_2,x_3] \cdot [y_1 + z_1 , y_2 + z_2 , y_3 + z_3] \\
&\quad = x_1(y_1 + z_1) + x_2(y_2 + z_2) + x_3(y_3 + z_3) \\
&\quad = x_1y_1 + x_1z_1 + x_2y_2 + x_2z_2 + x_3y_3 + x_3z_3 \\
&\quad = x_1y_1 + x_2y_2 + x_3y_3 + x_1z_1 + x_2z_2 + x_3z_3 \\
&\quad = \vec{X} \cdot \vec{Y} + \vec{X} \cdot \vec{Z} .
\end{aligned}$$

This is the right-hand side of the equation and the proof is complete.

(b) To prove $(t\vec{X}) \cdot \vec{Y} = t(\vec{X} \cdot \vec{Y})$.

The left-hand member is expanded to obtain

$$\begin{aligned}
&[tx_1,tx_2,tx_3] \cdot [y_1,y_2,y_3] \\
&\quad = tx_1y_1 + tx_2y_2 + tx_3y_3 \\
&\quad = t(x_1y_1 + x_2y_2 + x_3y_3) ,
\end{aligned}$$

which, by Theorem 8-3, is $t(\vec{X} \cdot \vec{Y})$.

Proof is complete.

Corollary. To prove that

$$\vec{X} \cdot (a\vec{Y} + b\vec{Z}) = a(\vec{X} \cdot \vec{Y}) + b(\vec{X} \cdot \vec{Z}) ,$$

we expand the left-hand member to obtain

$$\begin{aligned}
&[x_1,x_2,x_3] \cdot [ay_1 + bz_1 + ay_2 + bz_2 + ay_3 + bz_3] \\
&\quad = ax_1y_1 + bx_1z_1 + ax_2y_2 + bx_2z_2 + ax_3y_3 + bx_3z_3] \\
&\quad = a(x_1y_1 + x_2y_2 + x_3y_3) + b(x_1z_1 + x_2z_2 + x_3z_3) \\
&\quad = a(\vec{X} \cdot \vec{Y}) + b(\vec{X} \cdot \vec{Z}) \text{ , or the right-hand member.}
\end{aligned}$$

Hence, the proof is complete.

8-6 Although there are not many new ideas in this section, some of the arguments require close attention. The postulates and definition mentioned are from the SMSG Geometry.

We note that even though P and P_1 are in the plane M, $\overrightarrow{P} - \overrightarrow{P_1}$ denotes an origin-vector which does not lie in M unless M also contains the origin.

Example 3 through Example 6 are not essential to the development, but are included to show students that entire regions or their boundaries may be described concisely with vectors.

Exercises 8-6

1. (a) $7x - 3y + 5z = 15$

2. We assume the plane in question contains the given points in each of the following.

   (a) $2x - 3y + z - 14 = 0$

   (b) $2x - 4y + 7z + 69 = 0$

   (c) $3x - 5y + 4z - 50 = 0$

   (d) $x + y - 6z + 38 = 0$

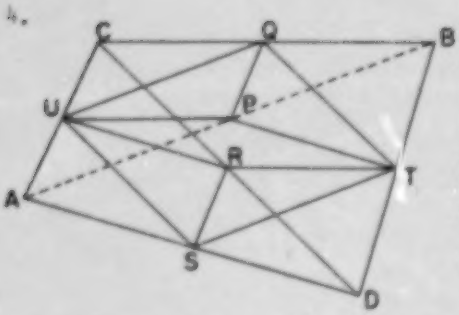
3. (a) $\frac{5}{\sqrt{14}}$ or $\frac{5\sqrt{14}}{14}$

   (b) $\frac{8}{\sqrt{38}}$ or $\frac{4\sqrt{38}}{19}$

   (c) if $d = 0 : 0$

   if $d \neq 0 : \frac{|d|}{\sqrt{a^2 + b^2 + c^2}}$

4\.



A,B,C, and D are the four vertices of the tetrahedron and P,Q,R,S,T and U are the midpoints of $\overline{AB}$,$\overline{BC}$,$\overline{CD}$,$\overline{DA}$,$\overline{BD}$, and $\overline{CA}$ respectively. Thus

$\vec{P} = \frac{1}{2}(\vec{A} + \vec{B})$ $\quad \vec{Q} = \frac{1}{2}(\vec{B} + \vec{C})$

$\vec{R} = \frac{1}{2}(\vec{C} + \vec{D})$ $\quad \vec{S} = \frac{1}{2}(\vec{D} + \vec{A})$

$\vec{T} = \frac{1}{2}(\vec{B} + \vec{D})$ $\quad \vec{U} = \frac{1}{2}(\vec{A} + \vec{C})$

(a) To show that $\overleftrightarrow{QS}$, $\overleftrightarrow{UT}$, and $\overleftrightarrow{PR}$ are concurrent. $\overleftrightarrow{QS}$ is represented by $x\vec{Q} + (1 - x)\vec{S}$ or $\frac{1}{2}((1 - x)\vec{A} + x\vec{B} + x\vec{C} + (1 - x)\vec{D})$. $\overleftrightarrow{UT}$ is represented by $y\vec{U} + (1 - y)\vec{T}$ or $\frac{1}{2}(y\vec{A} + (1 - y)\vec{B} + (1 - y)\vec{C} + y\vec{D})$. For these to intersect there must be x and y such that $(1 - x) = y$ and $(1 - y) = x$. But $x = y = \frac{1}{2}$ meets this condition so $\frac{1}{4}(\vec{A} + \vec{B} + \vec{C} + \vec{D})$ is on both $\overleftrightarrow{QS}$ and $\overleftrightarrow{UT}$. But

$$\frac{1}{4}(\vec{A} + \vec{B} + \vec{C} + \vec{D}) = \frac{1}{2}(\frac{1}{2}(\vec{A} + \vec{B}) + \frac{1}{2}(\vec{C} + \vec{D})) = \frac{1}{2}\vec{P} + (1 - \frac{1}{2})\vec{R}$$

which is on $\overleftrightarrow{PR}$. Hence the three lines are concurrent.

(b) We wish to show that QUST and PURT are parallelograms. First we must show that QUST and PURT are plane figures. However, from (a) we know that $\overleftrightarrow{QS}$ intersects $\overleftrightarrow{UT}$ and that $\overleftrightarrow{PR}$ intersects $\overleftrightarrow{UT}$, so we have coplanarity.

Then $\vec{Q} - \vec{U} = \frac{1}{2}(\vec{B} - \vec{A}) = \vec{T} - \vec{S}$, so $d(Q,U) = d(B,A)$,

and $\vec{Q} - \vec{T} = \frac{1}{2}(\vec{C} - \vec{D}) = \vec{U} - \vec{S}$, so $d(Q,T) = d(U,S)$.

Thus QUST has two pairs of opposite sides of equal length and is thus a parallelogram. We could not get from $\overleftrightarrow{QU} \| \overleftrightarrow{BA}$ and $\overleftrightarrow{TS} \| \overleftrightarrow{BA}$ to $\overleftrightarrow{QU} \| \overleftrightarrow{TS}$ without assuming or proving the theorem in solid geometry that for any lines a , b , and c , $a \| b$ and $b \| c$ imply $a \| c$. The proof that PURT is a parallelogram procedes similarly. $\overleftrightarrow{PR}$ intersects $\overleftrightarrow{UT}$, so P , U , R , and T are coplanar. We show that $\vec{P} - \vec{T} = \frac{1}{2}(\vec{A} - \vec{D}) = \vec{U} - \vec{R}$

and $\vec{P} - \vec{U} = \frac{1}{2}(\vec{B} - \vec{C}) = \vec{T} - \vec{R}$,

so that $d(P,T) = d(U,R)$ and $d(P,U) = d(R,T)$.
Hence PURT is a parallelogram.

(c) Since in (a) we show that $x = y = \frac{1}{2}$, the point of concurrency is the midpoint of the segments involved.

5. Let $\overrightarrow{D}$ be the normal vector from P_1 to M.

$$\overrightarrow{D} = [x - x_1, y - y_1, z - z_1] .$$

The unit vector normal to M is

$$\overrightarrow{n} = [\lambda, \mu, \nu] .$$

Then the distance from P_1 to M is found from

$$d = |\overrightarrow{n} \cdot \overrightarrow{D}| = |[\lambda, \mu, \nu] \cdot [x - x_1, y - y_1, z - z_1]|$$

$$= |\lambda x - \lambda x_1 + \mu y - \mu y_1 + \nu z - \nu z_1| = |\lambda x_1 + \mu y_1 + \nu z_1 - p| .$$

6. (a) $\overleftrightarrow{AB} = \{x : \overrightarrow{x} = [4, 9p - 7, -2p + 5]\}$

(b) $\overline{AB} = \{x : \overrightarrow{x} = [3 - 5p, 4 - p, 2 + p], 0 \leq p \leq 1\}$

(c) $\overrightarrow{AB} = \{x : \overrightarrow{x} = [3 - 5p, 4 - p, 2 + p], p \geq 0\}$

(d) $\overrightarrow{BA} = \{x : \overrightarrow{x} = [3 - 5p, 4 - p, 2 + p], p \leq 0\}$.

7. (a) Midpoint $\overrightarrow{M} = [3, 6, 7\frac{1}{2}]$

trisection points $\overrightarrow{T}_1 = [2, 4, 5]$ and $\overrightarrow{T}_2 = [4, 8, 10]$.

(b) $\overrightarrow{M} = [3\frac{1}{2}, -4\frac{1}{2}, 13]$

$\overrightarrow{T}_1 = [2\frac{1}{3}, -2\frac{1}{3}, 8\frac{2}{3}]$ and $\overrightarrow{T}_2 = [5\frac{2}{3}, -6\frac{2}{3}, 15]$.

(c) $\overrightarrow{M} = \frac{1}{2}[a_1 + b_1, a_2 + b_2, a_3 + b_3]$

$\overrightarrow{T}_1 = \frac{1}{3}[2a_1 + b_1, 2a_2 + b_2, 2a_3 + b_3]$ and

$\overrightarrow{T}_2 = \frac{1}{3}[a_1 + 2b_1, a_2 + 2b_2, a_3 + 2b_3]$.

8. (a) $\overrightarrow{X} = [0, 0, 0]$

(b) $\overrightarrow{X} = [\frac{2}{3}, 2\frac{1}{2}, -5\frac{1}{2}]$

(c) $\overrightarrow{X} = [1\frac{1}{4}, -\frac{3}{4}, 3\frac{1}{4}]$

9. (a) The triangular region is $\{Y : \overleftrightarrow{Y} = [1 + p - 2q + 2pq,$ $4 - p - 2q + 2pq, -2 + 3p + 6q - 6pq]$ where $0 \leq p \leq 1$ and $0 \leq q \leq 1\}$.

The interior is the same except $0 < p < 1$ and $0 < q < 1$.

The triangle is $\{Y : \overleftrightarrow{Y} = [1 + p - 2q + 2pq, 4 - p - 2q + 2pq,$ $-2 + 3p + 6q - 6pq]$ where $(p = 0$ and $0 \leq q \leq 1)$ or $(q = 0$ and $0 \leq p \leq 1)$ or $(q = 1$ and $0 \leq p \leq 1)\}$.

(b) $p = q = \frac{1}{2}$ in the above gives the desired point $[1,3,1]$. Hence $[1,3,1]$ must be an interior point.

(c) If $[-4,-5,-6]$ is in the triangular region, then $p - 2q + 2pq = -4$ and $4 - p - 2q + 2pq = -5$. If we solve this, we find $p = 2$ and hence $[-4,-5,-6]$ cannot be in the triangular region.

Review Exercises

1. $z = 5$

2. $x^2 = 25$

3. $x^2 = z^2.$

4. $y^2 + z^2 = 4$

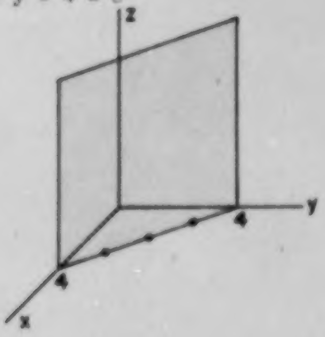
5. $x^2 + y^2 + z^2 = a^2$

6. $(x - 2)^2 + (y + 1)^2 + z^2 = r^2$

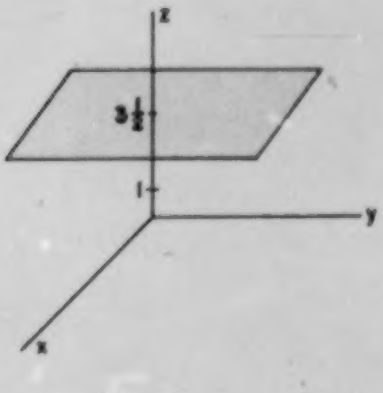
7. $(x - 1)^2 + (y - 2)^2 - 4z + 5 = 0$

8. $x + y + z - 6 = 0$

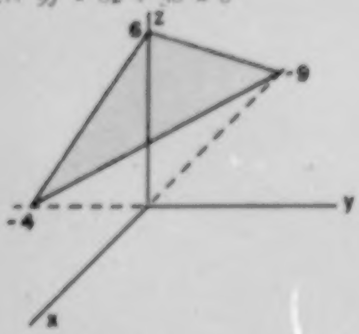
9. $x + y - 4 = 0$



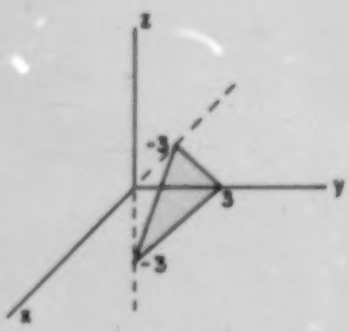
10. $2z - 7 = 0$



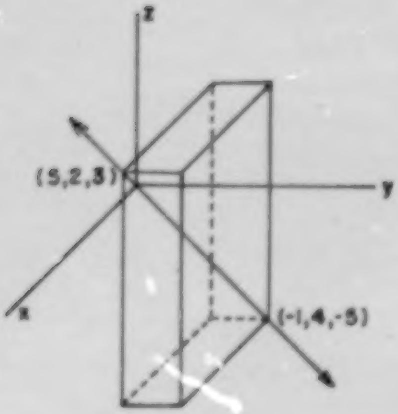
11. $4x + 9y - 6z + 36 = 0$



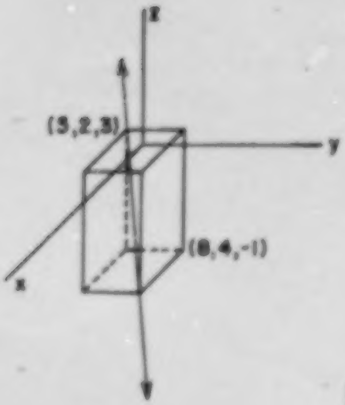
12. $x - y + z + 3 = 0$



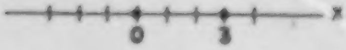
13. $x = 5 - 3t$
$y = 2 + t$
$z = 3 - 4t$

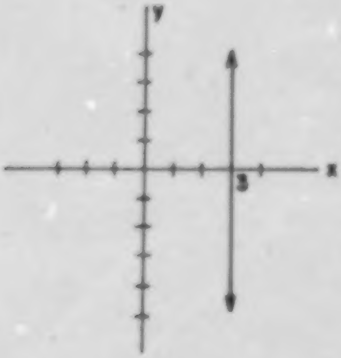


14. $\frac{x - 5}{-3} = \frac{y - 2}{-2} = \frac{z - 3}{4}$.



15.



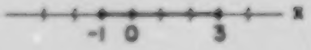


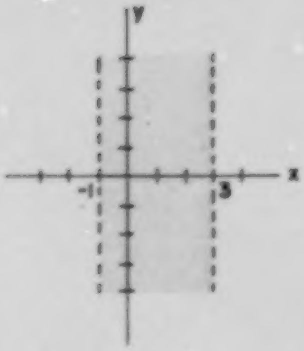
One-space: a point 3 units to the right of the origin.

2-space: a line perpendicular to the x-axis and 3 units to the right of the origin.

3-space: a plane perpendicular to the x-axis and 3 units in the positive direction from the origin.

16.



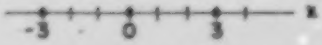


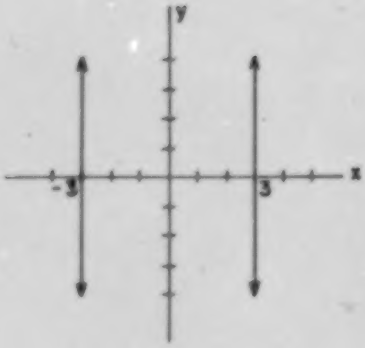
One space: a segment between, but not including, the points $x = -1$ and $x = 3$.

2-space: a portion of the xy-plane between, but not including, the lines $x = -1$ and $x = 3$.

3-space: a portion of space between, but not including, the planes $x = -1$ and $x = 3$. (It may be visualized as the path made by moving a plane parallel to the yz-plane.)

17.



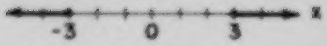


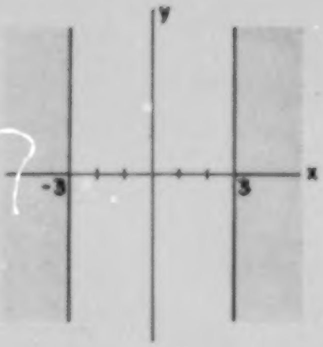
One-space: two points, $x = 3$ and $x = -3$.

2-space: two lines, $x = 3$ and $x = -3$.

3-space: two planes, $x = 3$ and $x = -3$.

18.



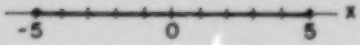


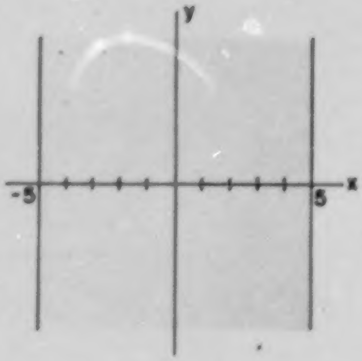
One-space: the portion of the x-axis to the right of and including $x = 3$, and to the left of and including $x = -3$.

2-space: the portion of the plane to the right of and including the line $x = 3$, and to the left of and including $x = -3$.

3-space: the portion of space beyond (in the positive direction) and including the plane $x = 3$, and the portion beyond (in the negative direction) and including the plane $x = -3$.

19.



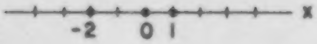


One-space: a segment between and including the points $x = -5$ and $x = 5$.

2-space: a portion of the xy-plane between and including the lines $x = -5$ and $x = 5$.

3-space: a portion of space between and including the planes $x = -5$ and $x = 5$.

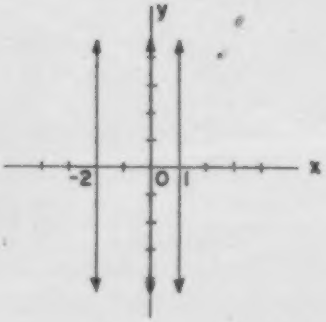
20.



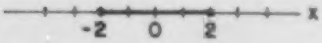
One-space: the points $x = -2$, $x = 0$, and $x = 1$.

2-space: the lines $x = -2$, $x = 0$, and $x = 1$.

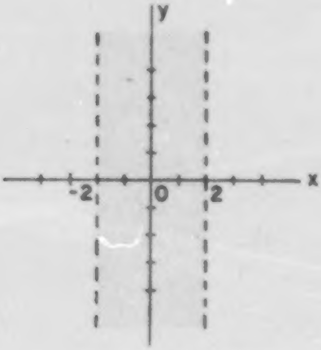
3-space: the planes $x = -2$, $x = 0$, and $x = 1$.



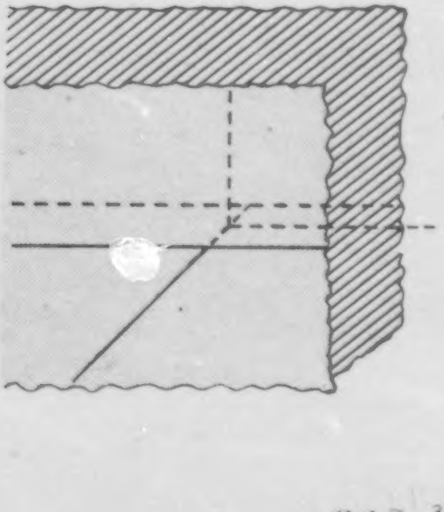
21. $R_1 = \{(x,y) : |x| < 2\}$



One-space: a segment between but excluding the points $x = -2$ and $x = 2$.

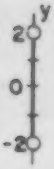


2-space: a portion of space between but excluding the lines $x = -2$ and $x = 2$.

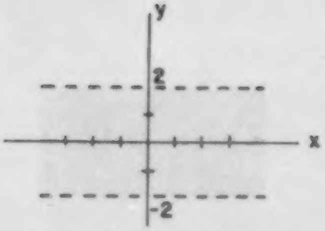


3-space: a portion of space between but excluding the planes $x = -2$ and $x = 2$.

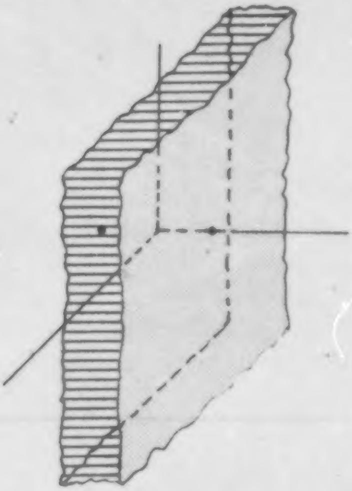
$R_2 = \{(x,y) : |y| < 2\}$



One-space (if we choose the y-axis): a segment between but excluding the points $y = -2$ and $y = 2$.



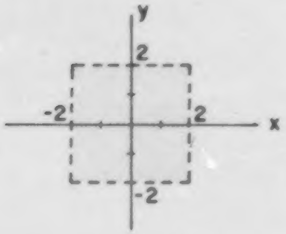
2-space: a portion of the plane between but excluding the lines $y = -2$ and $y = 2$.



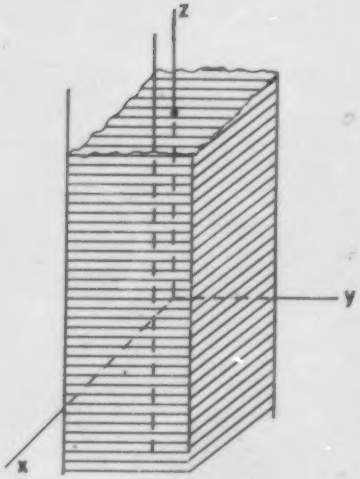
3-space: a portion of space between but excluding the planes $y = -2$ and $y = 2$.

$R_3 = R_1 \cap R_2$

One-space: R_3 is the null set



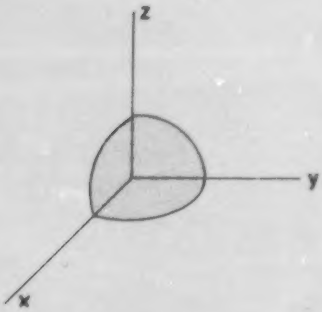
2-space: the interior of the square bounded by $x = -2$, $x = 2$ $y = -2$, $y = 2$.



3-space: the interior of the prism bounded by the plane $x = -2$, $x = 2$, $y = -2$, $y = 2$.

22. If, in Exercise 21, $<$ is changed to $\leq$ the graphs include the points, lines, and planes which are only boundaries and are not included in Exercise 21. $R_1 \cup R_2$ is the union of the first two graphs and would include all points in R_1 or R_2.

23.



The graph of $x^2 + y^2 + z^2 \leq 1$ represents a sphere with the center at the origin and all the points within the sphere. (Only a portion of the graph is shown.) If $\leq$ is changed to $<$, the graph includes only the points within the sphere.

24.

	Distance	A	B	C	D	O
(a)	M_1	0	$\frac{5}{\sqrt{14}}$	$\frac{2}{\sqrt{14}}$	$\frac{17}{\sqrt{14}}$	$\frac{4}{\sqrt{14}}$
(b)	M_2	$\frac{4}{\sqrt{14}}$	$\frac{1}{\sqrt{14}}$	$\frac{2}{\sqrt{14}}$	$\frac{7}{\sqrt{14}}$	$\frac{3}{\sqrt{14}}$
(c)	M_3	$\frac{7}{\sqrt{14}}$	$\frac{13}{\sqrt{14}}$	$\frac{13}{\sqrt{14}}$	$\frac{9}{\sqrt{14}}$	$\frac{2}{\sqrt{14}}$
(d)	M_4	$\frac{5}{\sqrt{3}}$	$\frac{5}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$\frac{3}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$

25. (a) $\frac{x + 11}{5} = \frac{y}{1} = \frac{z - 18}{-7}$

(b) $\frac{x}{1} = \frac{y - 2}{1} = \frac{z - 2}{1}$

(c) $\frac{x - 1}{4} = \frac{y - 2}{3} = \frac{z}{1}$

(d) $\frac{x}{-1} = \frac{y - 5}{11} = \frac{z - 4}{7}$

(e) $\frac{x - 2}{3} = \frac{y - 3}{5} = \frac{z}{-2}$

(f) $\frac{x - 2}{5} = \frac{y - 1}{2} = \frac{z - 2}{3}$

26. (a) $[x,y,z] = [-11,0,18] + t[5,1,-7]$

(b) $[x,y,z] = [0,2,2] + t[1,1,1]$

(c) $[x,y,z] = [1,2,0] + t[4,-3,1]$

(d) $[x,y,z] = [0,5,4] + t[-1,11,7]$

(e) $[x,y,z] = [2,3,0] + t[3,5,-2]$

(f) $[x,y,z] = [2,1,2] + t[5,2,3]$

27. One method is to find direction numbers; for $\overleftrightarrow{AB}$ and $\overleftrightarrow{DC}$ direction numbers are $(-2,2,6)$, hence they are parallel. Direction numbers for $\overleftrightarrow{BC}$ and $\overleftrightarrow{AD}$ are $(5,-4,-4)$.

28. Parametric equations of the medians

from A , $x = -t_1$, $y = 3t_1$, $z = -t_1$.

from B , $x = 2 - 4t_2$, $y = 4 - 3t_2$, $z = 6 - 10t_2$.

from C , $x = -4 + 5t_3$, $y = 2$, $z = -8 + 11t_3$.

The medians are concurrent in the point $(-\frac{2}{3}, 2, -\frac{2}{3})$.

29. $a = 7$

30. $a = -3$

Teachers' Commentary

Chapter 9

QUADRIC SURFACES

Since many of the students who study this course are likely candidates for college-level mathematics, this chapter has been included to give the students help in vizualizing and handling the types of objects they will encounter in later courses. Our sights are particularly set on the calculus. Even in the elementary applications of calculus, one encounters solid, or 3-space, figures of the non-rectangular variety. It might aid some students if you were to roughly describe calculus as a sort of super-algebra which enables us to handle areas of objects with curved sides and volumes of objects with curved surfaces. Ordinary algebra is generally powerless with objects which do not have straight sides or flat surfaces. Of course, this would not be a complete description of the power of the calculus, but some such discussion could be used to motivate the study of this chapter.

Except for the Challenge Problems and an occasional natural extension in the Exercises, we have limited the discussion to very simple forms of surfaces. In most cases the origin is chosen as the center of the figure, and the axes or elements of the figure are oriented along some coordinate axis. This simplifies the drawing techniques and the algebraic manipulations. More complicated forms are obtained by simple extensions.

Quadrics hold much the same place among surfaces that the conics occupy among curves and, next to the plane, are by far the most important types of surfaces.

It is usual to identify nine species of quadric surfaces, but in order to keep life in this chapter simple we have presented only six of these in the students' text. The other three types (numbers 5 , 7 , and 8 in the list below) appear in Challenge Problems. For completeness we list the nine species with an example equation for each.

1. Ellipsoid. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

2. Hyperboloid of one sheet. $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$.

3. Hyperboloid of two sheets. $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$.

4. Elliptic paraboloid. $\frac{x^2}{a^2} + \frac{y^2}{b^2} - cz = 0$.

5. Hyperbolic paraboloid. $\frac{x^2}{a^2} - \frac{y^2}{b^2} - cz = 0$.

6. Elliptic cylinder. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

7. Parabolic cylinder. $x^2 - cz = 0$.

8. Hyperbolic cylinder. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

9. Elliptic cone. $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$.

It is of considerable interest to note how the surface changes when the equation is altered by changing a sign, or by changing the power of a variable, or by changing the value of one or more of the constants. However, we felt that this material would make the course too long, so we somewhat reluctantly restricted the quadrics to the six simplest ones.

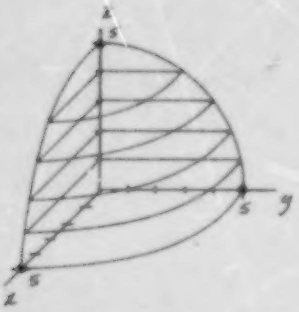
For the sake of variety, some of the spheres in Exercises 9-2 have been located away from the origin. If the coefficients of x^2 , y^2 , and z^2 are all equal, then the quadratic represents a sphere. One may then complete the square for each variable and obtain an equation of the form

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2 .$$

The point (x_0,y_0,z_0) is the center and r is the radius.

Exercises 9-2

1.



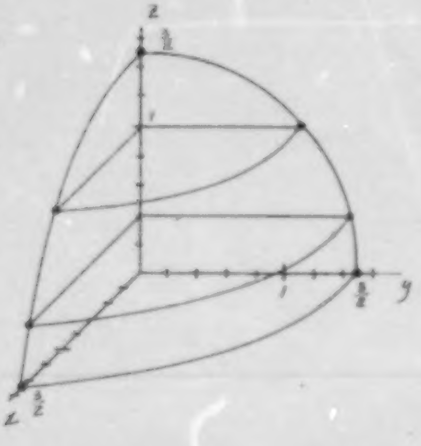
A sphere.

Center: origin

Radius: 5

All traces are circles.

2.


z
$\frac{3}{2}$
1
1
$\frac{3}{2}$
y
$\frac{3}{2}$
x

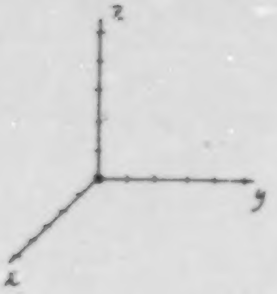
A sphere.

Center: origin

Radius: $\frac{3}{2}$

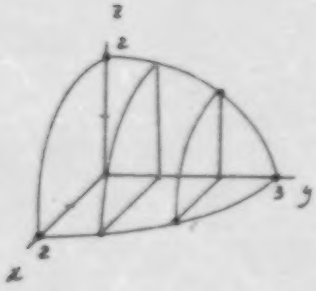
All traces are circles.

3.



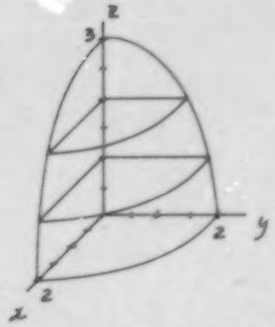
A point-sphere.
Center: origin
Radius: 0
The origin is the only point of the locus.

4.



A prolate spheroid.
Center: origin
The xy- and yz-traces are ellipses.
The xz-trace is a circle.

5.



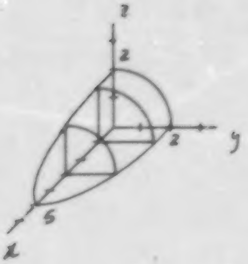
A prolate spheroid.

Center: origin

The xz- and yz-traces are ellipses.

The xy-trace is a circle.

6.



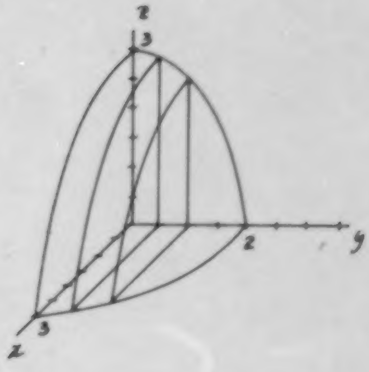
A prolate spheroid.

Center: origin

The xy- and xz-traces are ellipses.

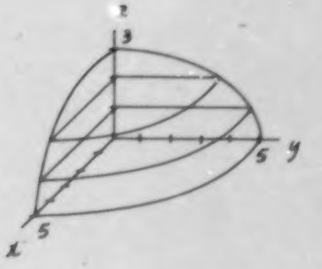
The yz-trace is a circle.

7.



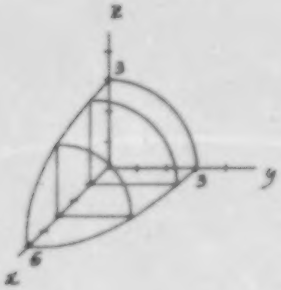
An oblate spheroid.
Center: origin
The xy- and yz-traces are ellipses.
The xz-trace is a circle.

8.



An oblate spheroid.
Center: origin
The xz- and yz-traces are ellipses.
The xy-trace is a circle.

9.



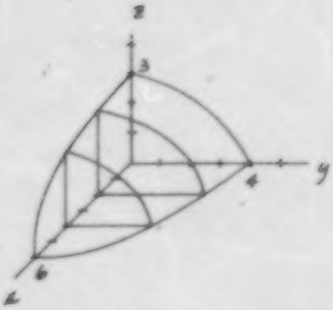
A prolate spheroid.

Center: origin

The xy- and xz-traces are ellipses.

The yz-trace is a circle.

10.

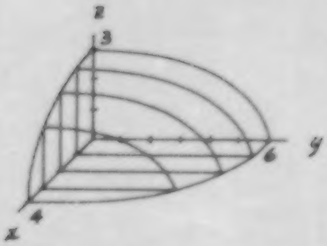


An ellipsoid

Center: origin

All traces are ellipses.

11.

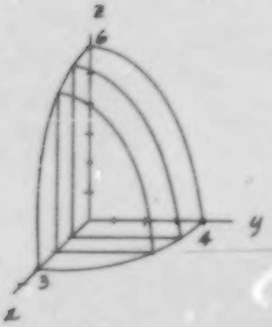


An ellipsoid.

Center: origin

All traces are ellipses.

12.



An ellipsoid.

Center: origin

All traces are ellipses.

13. $\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} = r$,

or $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$.

14. $x^2 - 2xx_0 + x_0^2 + y^2 - 2yy_0 + y_0^2 + z^2 - 2zz_0 + z_0^2 = r^2$,

or $x^2 + y^2 + z^2 - (2x_0)x - (2y_0)y - (2z_0)z + (x_0^2 + y_0^2 + z_0^2 - r^2) = 0$.

Since (x_0,y_0,z_0) represents any point and $r > 0$, the given equation represents a sphere with radius r and center at (x_0,y_0,z_0) .

15. (a) $x^2 + y^2 + z^2 - 4x - 2y - 6z - 11 = 0$.

(b) $x^2 + y^2 + z^2 + 2y - 4z + 1 = 0$.

(c) $x^2 + y^2 + z^2 - 2x - 6y + 4z + 12 = 0$.

(d) $x^2 + y^2 + z^2 - \frac{2}{3}x + 2y - z + \frac{13}{36} = 0$,

or $36x^2 + 36y^2 + 36z^2 - 24x + 72y - 36z + 13 = 0$.

(e) $x^2 + y^2 + z^2 - x - \frac{1}{2}y + z + \frac{5}{16} = 0$,

or $16x^2 + 16y^2 + 16z^2 - 16x - 8y + 16z + 5 = 0$.

(f) $x^2 + y^2 + z^2 - 3x + y - 5z - .25 = 0$

or $4x^2 + 4y^2 + 4z^2 - 12x + 4y - 20z - 1 = 0$.

16. (a) Center: origin. Radius: $\sqrt{3}$.

(b) Center: $(1,-2,3)$. Radius: 2 .

(c) Center: $(0,2,-1)$. Radius: 5 .

(d) Center: $(-3,4,-7)$. Radius: $\sqrt{2}$.

(e) Center: $(-2,3,0)$. Radius: 0 . (a point-sphere.)

(f) Not a sphere.

(g) Center: $(\frac{1}{2}, \frac{2}{3}, -1)$. Radius: $\frac{1}{2}$.

(h) Center: $(\frac{3}{4}, 2, -\frac{1}{2})$. Radius: $\frac{3}{2}$.

17. Center: $(0,1,5)$. Radius: $\sqrt{6}$

Equation of the sphere: $x^2 + y^2 + z^2 - 2y - 10z + 20 = 0$.

18. $\frac{x^2}{9} + \frac{y^2}{49} + \frac{z^2}{25} = 1$, or $1225x^2 + 225y^2 + 441z^2 = 11025$.

Challenge Problems

1. $\frac{(x - 3)^2}{36} + \frac{(y + 1)^2}{16} + \frac{(z - 2)^2}{144} = 1$,

   or $4x^2 + 9y^2 + z^2 - 24x + 18y - 4z - 95 = 0$.

2. Substituting the coordinates of the four points into the equation of general form, $x^2 + y^2 + z^2 + Dx + Ey + Fz + G = 0$, results in

$$\begin{array}{ll}
(1) & 3E + F + G = -10 \; , \\
(2) & -2D + 2F + G = -8 \; , \\
(3) & D + E + 4F + G = -18 \; , \\
(4) & -3D + 3E + 2F + G = -22 \; .
\end{array}$$

From these we obtain

$$\begin{array}{lll}
 & (2) & -2D + 2F + G = -8 \; , \\
(5) & (1) - (4) & 3D - F = 12 \; , \\
(6) & 3\cdot(3) - (1) & 3D + 11F + 2G = -44 \; .
\end{array}$$

Then we have

$$\begin{array}{lll}
(7) & (6) - 2\cdot(2) & 7D + 7F = -28 \; , \\
(8) & 7\cdot(5) & 21D - 7F = 84 \; , \text{ and} \\
(9) & (7) + (8) & 28D = 56 \; .
\end{array}$$

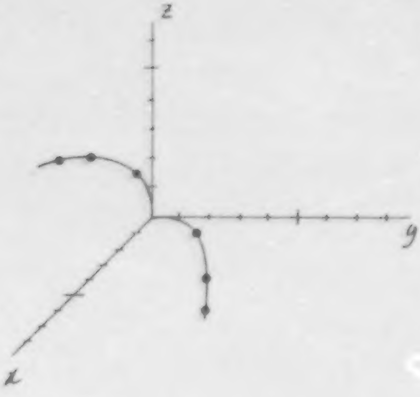
Therefore, $D = 2$, $F = -6$, $G = 8$, $E = -4$, and the equation is $x^2 + y^2 + z^2 + 2x - 4y - 6z + 8 = 0$. This can be written $(x + 1)^2 + (y - 2)^2 + (z - 3)^2 = 6$, showing the center and radius of the sphere.

As an alternate method, find the center of the sphere as the intersection of the perpendicular bisecting planes determined by pairs of the given points. The radius may then be found as the distance between the center and one of the given points.

Four points determine a sphere if the points are not coplanar and if no three points are collinear.

Exercises 9-3

1.



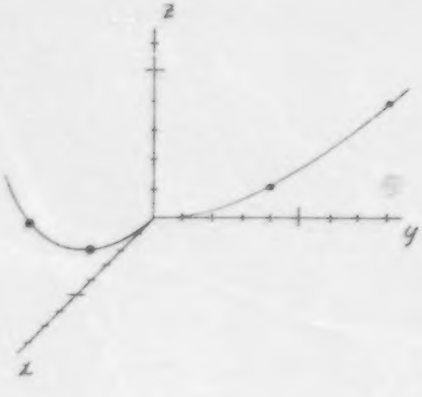
A paraboloid of revolution.

Axis: x-axis

Vertex: origin

The xy- and xz-traces are parabolas. Sections parallel to the yz-plane are circles. (These sections are not drawn because they interfere with other parts of the figure.)

2.



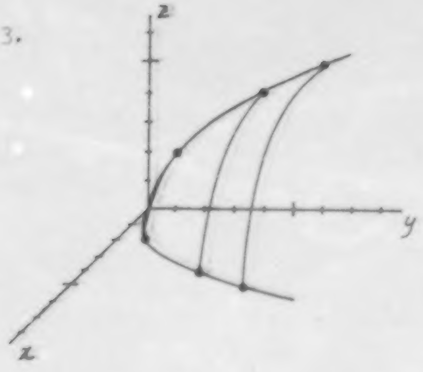
A paraboloid of revolution.

Axis: z-axis

Vertex: origin

The xz- and yz-traces are parabolas. Sections parallel to the xy-plane are circles (not shown).

3.



A paraboloid of revolution.

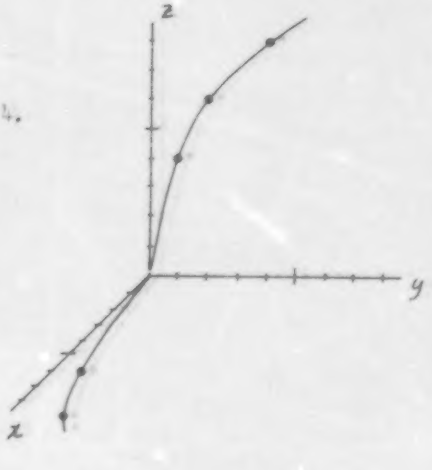
Axis: y-axis

Vertex: origin

The xy- and yz-traces are parabolas.

Sections parallel to the xz-plane are circles.

4.



An elliptic paraboloid.

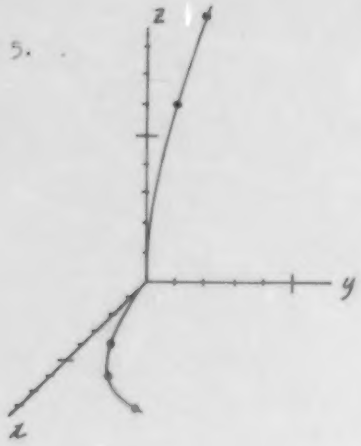
Axis: y-axis

Vertex: origin

The xy- and yz-traces are parabolas.

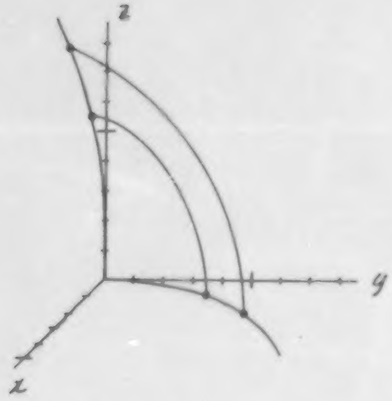
Sections parallel to the xz-plane are ellipses (not shown).

5.



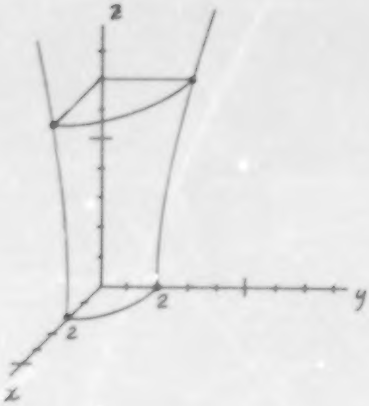
An elliptic paraboloid.
Axis: y-axis
Vertex: origin
The xy- and yz-traces are parabolas.
Sections parallel to the xz-plane
are ellipses (not shown).

6.



An elliptic paraboloid.
Axis: x-axis
Vertex: origin
The xy- and xz-traces are parabolas.
Sections parallel to the yz-plane
are ellipses.

7.



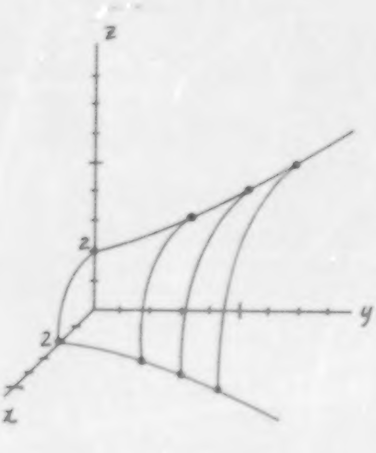
A hyperboloid of revolution (one sheet).

Axis: z-axis

The xy-trace is a circle of radius 2 .

The xz- and yz-traces are hyperbolas.

8.



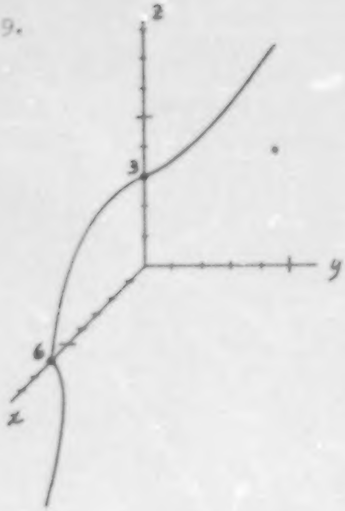
A hyperboloid of revolution (one sheet).

Axis: y-axis

The xz-trace is a circle of radius 2 .

The xy- and yz-traces are hyperbolas.

9.

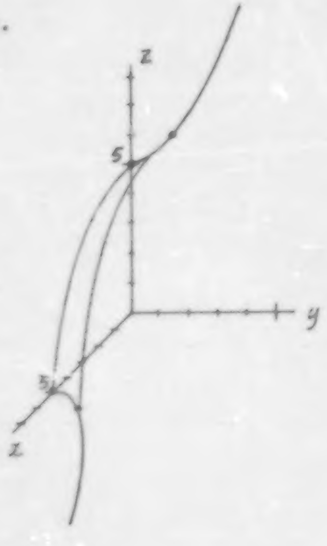


An elliptic hyperboloid (one sheet).

The xz-trace is an ellipse

The xy- and yz-traces are hyperbolas.

10.



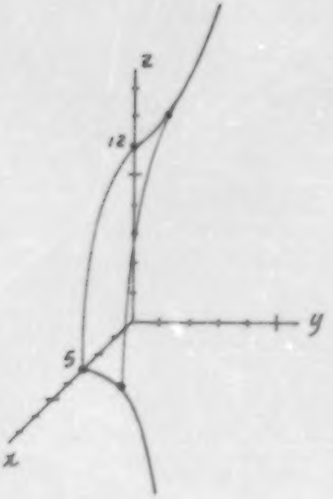
A hyperboloid of revolution (one sheet).

Axis: y-axis

The xz-trace is a circle of radius 5 .

The xy- and yz-traces are hyperbolas.

11.

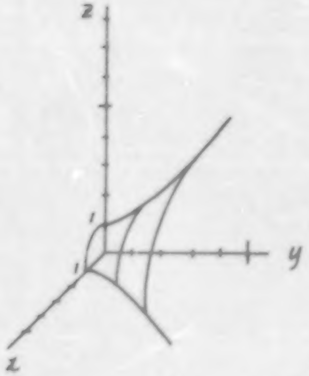


An elliptic hyperboloid (one sheet).

The xz-trace is an ellipse.

The xy- and yz-traces are hyperbolas.

12.



A hyperboloid of revolution (one sheet).

Axis: y-axis

The xz-trace is a circle of radius 1 .

The xy- and yz-traces are hyperbolas.

13. In Section 7-7 we have $e = \frac{\sqrt{a^2 - b^2}}{a} < 1$ as the eccentricity of an ellipse. Applying this to the equation of the xy-trace of Equation (3), $\frac{x^2}{4} + \frac{y^2}{9} = 1$, we find $e = \frac{\sqrt{5}}{3}$. For any section of the hyperboloid parallel to the xy-plane, say when $z = k$, we have

$\frac{x^2}{4} + \frac{y^2}{9} = 1 + \frac{k^2}{25}$. Let $1 + \frac{k^2}{25} = q^2$, $q > 0$, the equation then becomes

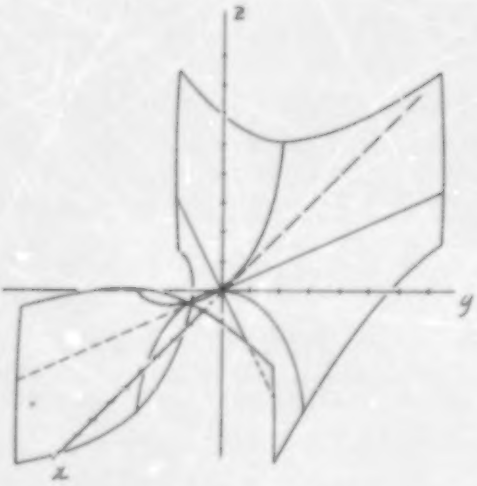
$\frac{x^2}{4q^2} + \frac{y^2}{9q^2} = 1$. Evaluating the eccentricity gives us

$$e = \frac{\sqrt{9q^2 - 4q^2}}{3q} = \frac{\sqrt{5}}{3}.$$

363 The Challenge Problems which follow contain work on hyperbolic paraboloids, which were omitted from the basic text. An aid to identifying several of these surfaces is in their names. The first part of the name (an adjective) indicates the kind of sections we find parallel to one coordinate plane; the second word (a noun) indicates the type of sections which are parallel to the other two coordinate planes.

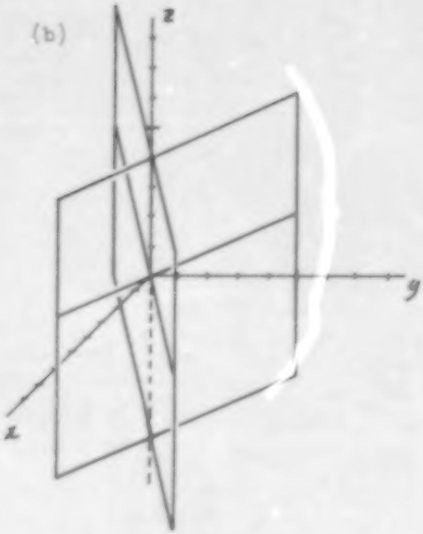
Challenge Problems

1. (a)



It will be noted that these "saddle shapes" are difficult to sketch. The yz-trace and the sections parallel to it are parabolas opening downward. The xz-trace and the sections parallel to it are parabolas opening upward. The sections parallel to the xy-plane are hyperbolas, which degenerate to a pair of intersecting lines as the xy-trace.

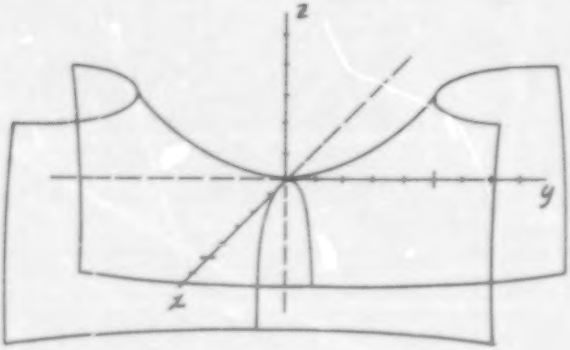
(b)



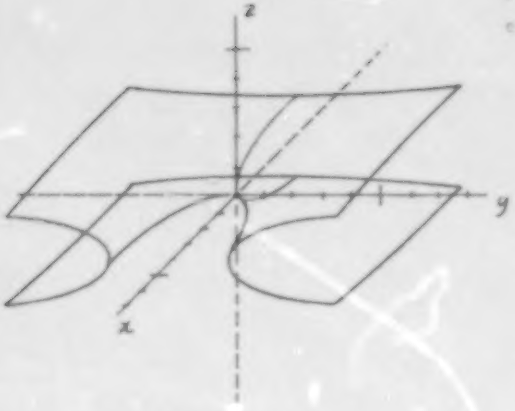
The planes shown in part (b) intersect on the z-axis and are determined by this axis and the xy-trace. These planes serve in an asymptotic capacity with respect to the hyperbolic paraboloid. A section parallel to the xy-plane (a hyperbola) will have as asymptotes the section of these two planes formed by the horizontal cutting plane.

If students encounter difficulty vizualizing this surface, have them look at the region between two knuckles of a clenched fist.

2. The sections parallel to the xz-plane are parabolas opening downward. The sections parallel to the yz-plane are parabolas opening upward.
The sections parallel to the xy-plane are hyperbolas, except for the xy-trace which is a pair of intersecting lines.



3\.

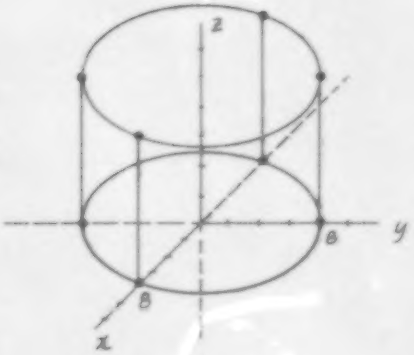


The sections parallel to the xy- and xz-planes are parabolas; the sections parallel to the yz-plane are hyperbolas, except for the degenerate yz-trace.

(6) A cylinder need not be "round". Any curve (or line) may be a directrix. We have purposely included in the basic text only closed cylinders. Other types, including sinusoidal ones, are found in the Challenge Problems.

Exercises 9-4

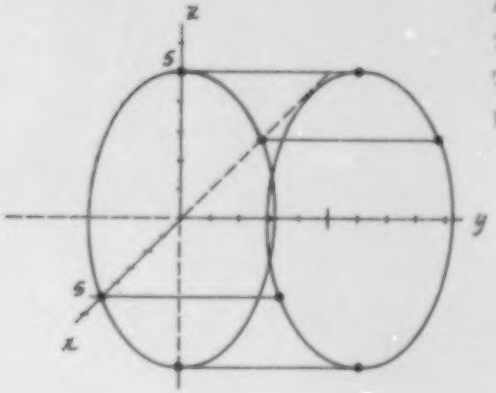
1\.



Axis of revolution: z-axis

The xy-trace and sections parallel to the xy-plane are circles of radius 8 . The xz- and yz-traces are lines parallel to the z-axis.

2.



Axis of revolution: y-axis

The xz-trace is a circle of radius 5 .

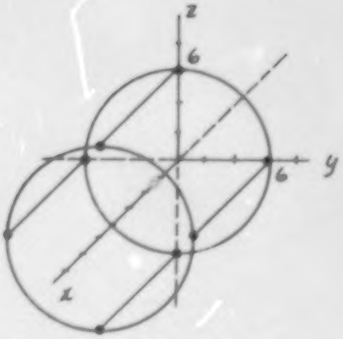
The xy- and yz-traces are lines
parallel to the y-axis.

3.

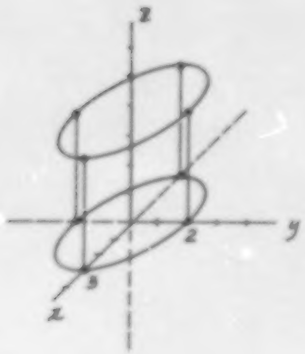
Axis of revolution: x-axis

The yz-trace is a circle of radius 6 .

The xy- and xz-traces are lines
parallel to the x-axis.



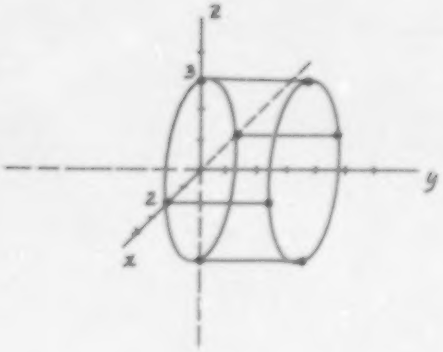
4.



The xy-trace (and sections parallel to it) is an ellipse.

The xz- and yz-traces are lines parallel to the z-axis.

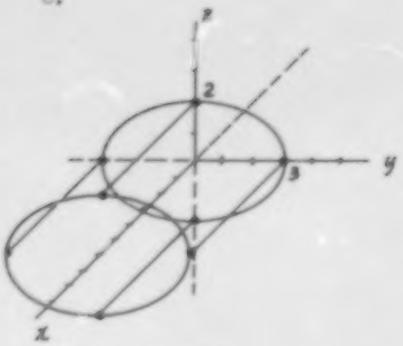
5.



The xz-trace is an ellipse.

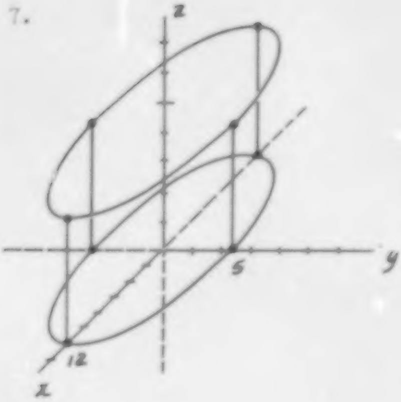
The xy- and yz-traces are lines parallel to the y-axis.

6.



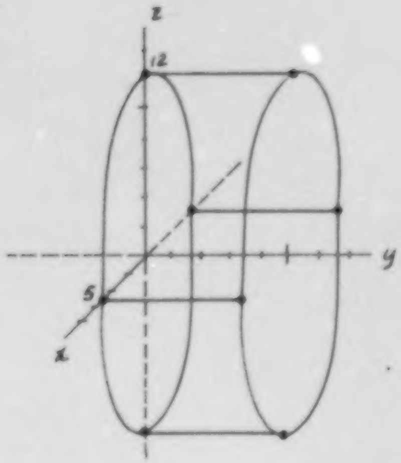
The yz-trace is an ellipse. The xy- and yz-traces are lines parallel to the x-axis.

7.



The xy-trace is an ellipse. The xz- and yz-traces are lines parallel to the z-axis.

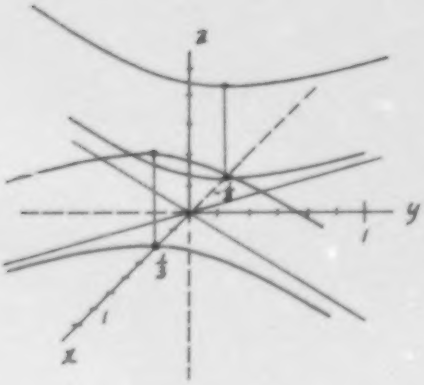
8.



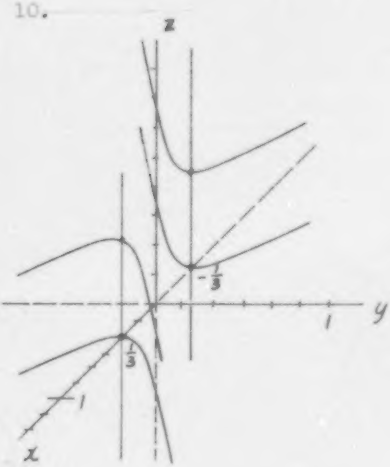
The xz-trace is an ellipse.
The xy- and yz-traces are lines parallel to the y-axis.

9.

The xy-trace (and sections parallel to it) is a hyperbola.
The xz-trace is a pair of lines parallel to the z-axis.
There is no yz-trace, but planes parallel to the yz-plane which intersect the surface cut off lines parallel to the z-axis.



10.



The xy-trace is a hyperbola.
The xz-trace is a pair of lines parallel to the z-axis.
There is no yz-trace, but planes parallel to the yz-plane which intersect the surface cut off lines parallel to the z-axis.

11. (a) $y^2 + z^2 = 81$.

(b) $x^2 + z^2 = 36$.

(c) $x^2 + y^2 = 16$.

12. (a) $y^2 + z^2 = 9$.

(b) $x^2 + z^2 = 25$.

(c) $x^2 + y^2 = 100$.

13. $x^2 + z^2 = 100$.

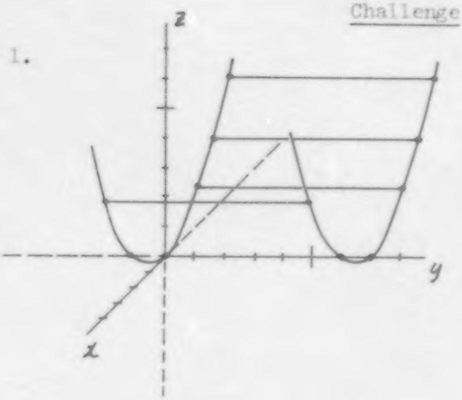
14. $y^2 + z^2 = 144$.

15. $x^2 + z^2 = 4$.

16. $25y^2 + 4z^2 = 100$.

Challenge Problems

1.

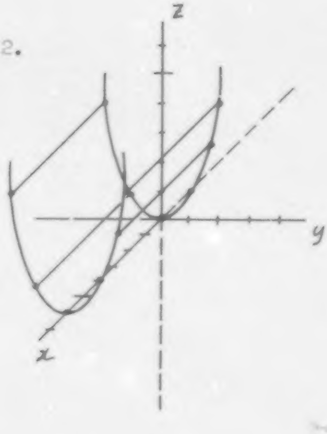


A parabolic cylinder.

The elements are parallel to the y-axis.

The directrix is a parabola.

2.

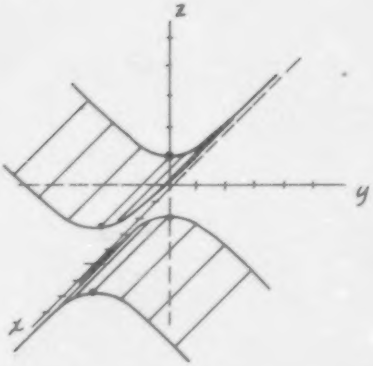


A parabolic cylinder.

The elements are parallel to the x-axis.

The directrix is a parabola.

3.

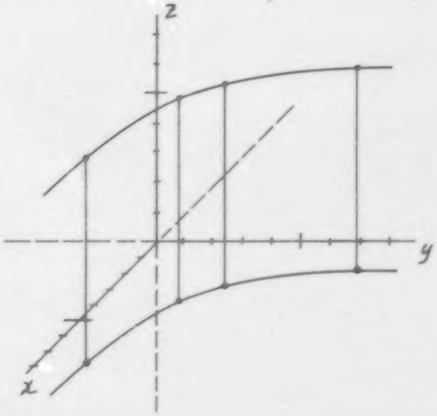


A hyperbolic cylinder.

The elements are parallel to the x-axis.

The directix is a hyperbola.

4.

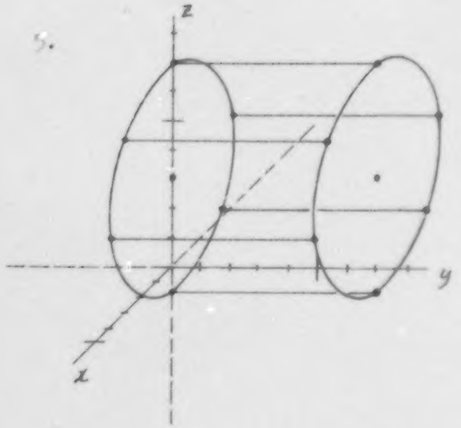


A hyperbolic cylinder (only one branch shown).

The elements are parallel to the z-axis.

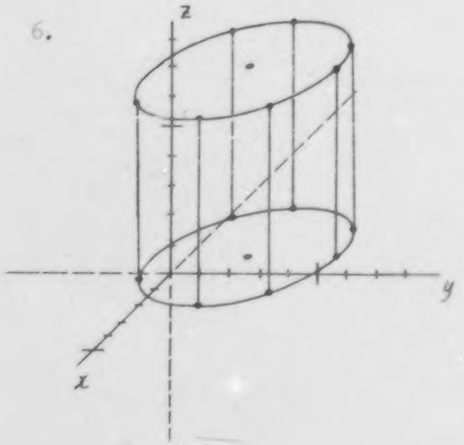
The directrix is an equilateral hyperbola.

5.



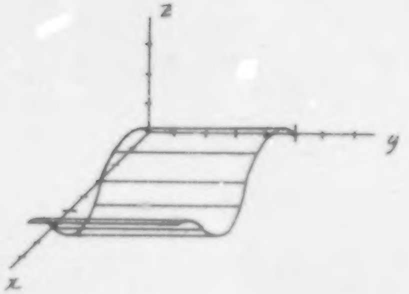
A circular cylinder.
The elements are parallel to the y-axis.
A directrix is a circle of radius 4 having its center at (0,0,3) .

6.



A circular cylinder.
The elements are parallel to the z-axis.
A directrix is a circle of radius 3 having its center at (-1,2,0) .

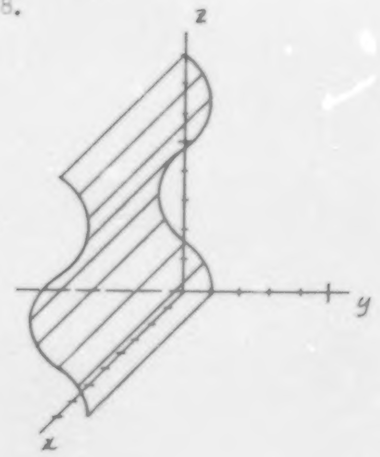
7.



The elements are parallel to the y-axis.

The directrix is a sine curve.

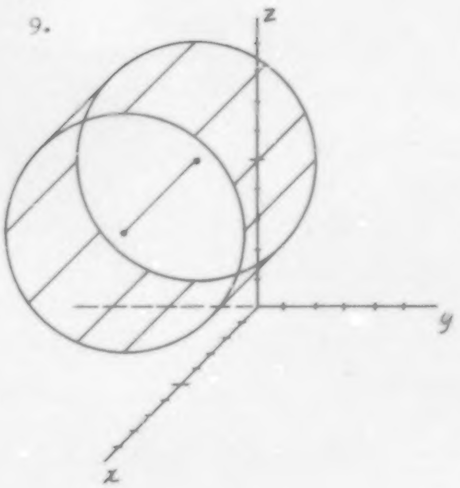
8.



The elements are parallel to the x-axis.

The directrix is a cosine (or displaced sine) curve.

9.

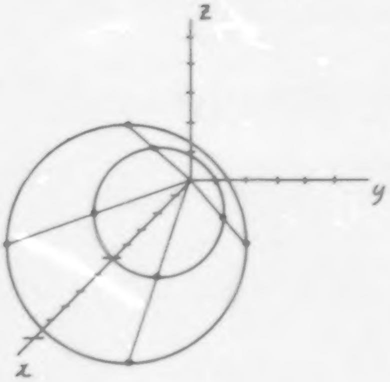


$(y + 2)^2 + (z - 5)^2 = 16$, or

$y^2 + z^2 + 4y - 10z + 13 = 0$.

Exercises 9-5

1.

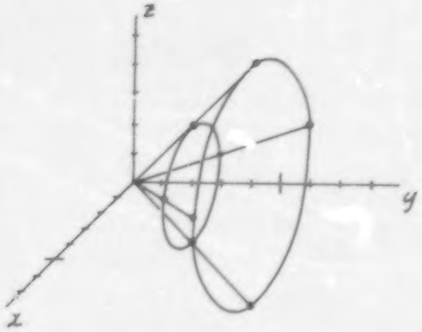


Axis: x-axis

Intercept: origin

Sections parallel to the yz-plane are circles. (One nappe only is shown.)

2.

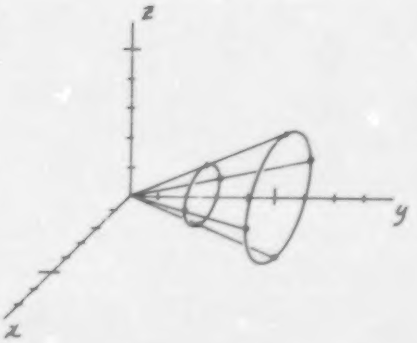


Axis: y-axis

Intercept: origin

Sections parallel to the xz-plane are circles.

3.

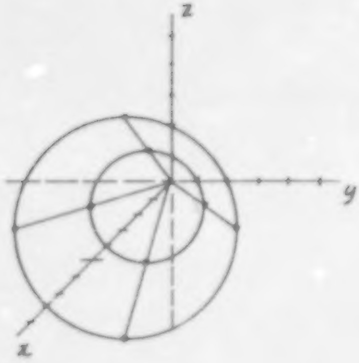


Axis: y-axis

Intercept: origin

Sections parallel to the xz-plane are circles.

4.

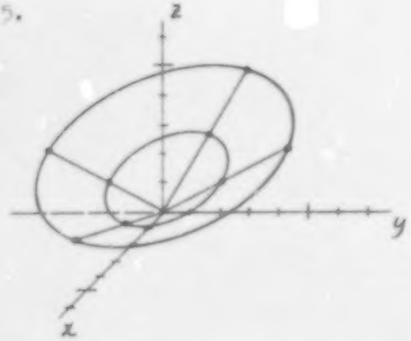


Axis: x-axis

Intercept: origin

Sections parallel to the yz-plane are circles.

5.

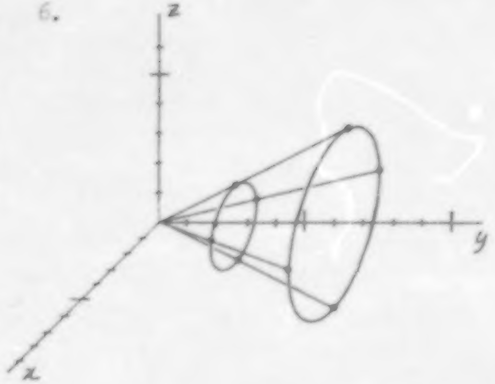


Axis: z-axis

Intercept: origin

Sections parallel to the xy-plane are ellipses.

6.



Axis: y-axis

Intercept: origin

Sections parallel to the xz-axis are ellipses.

7. $x^2 - 4y^2 + z^2 = 0$.

8. $-4x^2 + 4y^2 + 9z^2 = 0$.

9. $16x^2 + 16y^2 - 9z^2 = 0$.

10. $225x^2 - 16y^2 + 25z^2 = 0$.

11. The section in the plane $y = 1$ has the equation $\frac{x^2}{4} + \frac{z^2}{9} = 1$. The eccentricity of this ellipse is $e = \frac{\sqrt{9 - 4}}{3} = \frac{\sqrt{5}}{3}$. For any section of the cone parallel to the xz-plane, say when $y = k$ we have $\frac{x^2}{4} + \frac{z^2}{9} = k^2$, or $\frac{x^2}{4k^2} + \frac{z^2}{9k^2} = 1$. Evaluating the eccentricity gives us $e = \frac{\sqrt{9k^2 - 4k^2}}{3k} = \frac{\sqrt{5}}{3}$.

Challenge Problems

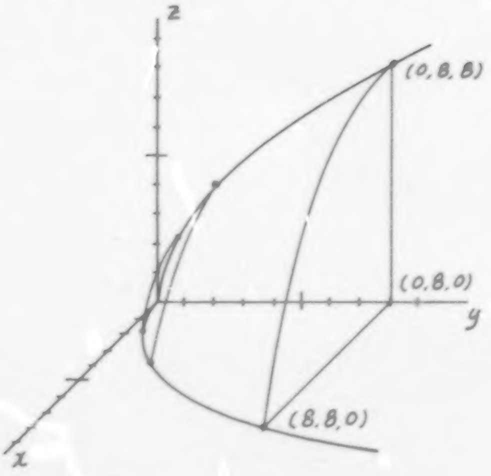
1. Since $a = 6$, $e = \frac{4}{6} = \frac{\sqrt{36 - 20}}{6}$; therefore, the ellipse in the plane $x = 1$ either has equation $\frac{y^2}{36} + \frac{z^2}{20} = 1$ or has equation $\frac{y^2}{20} + \frac{z^2}{36} = 1$. The cone is either $-180x^2 + 5y^2 + 9z^2 = 0$ or $-180x^2 + 9y^2 + 5z^2 = 0$.

2. Since $a = 8$, $e = \frac{4}{8} = \frac{\sqrt{64 - 48}}{8}$; therefore, the ellipse in the plane $z = 2$ either has equation $\frac{x^2}{16} + \frac{y^2}{12} = 1$ or has equation $\frac{x^2}{12} + \frac{y^2}{16} = 1$. The cone is either $4x^2 + 3y^2 - 48z^2 = 0$ or $3x^2 + 4y^2 - 48z^2 = 0$.

368 An interesting oral exercise might be interposed in this section. Have the students try to describe the surface generated by revolving about an axis of symmetry the printed capital form of certain letters of the English alphabet.

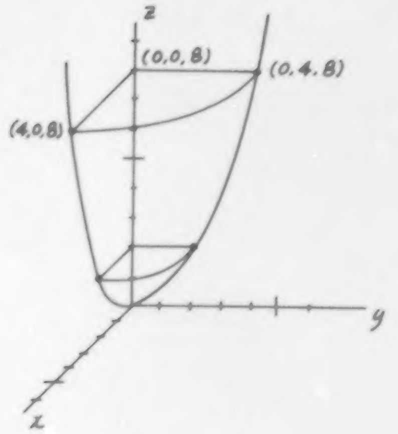
Exercises 9-6

1.



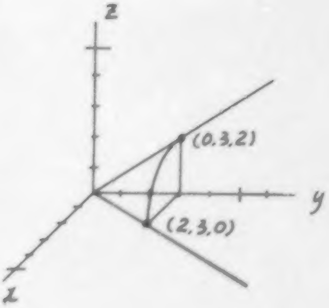
$$x^2 + z^2 = 8y .$$

2.


z
(0,0,8)
(0,4,8)
(4,0,8)
y
x

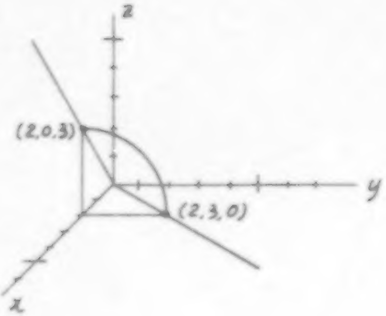
$x^2 + y^2 = 2z$.

3.



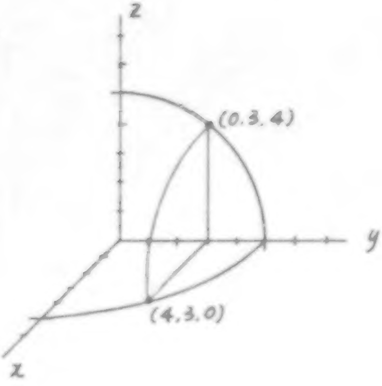
$9x^2 - 4y^2 + 9z^2 = 0$.

4.



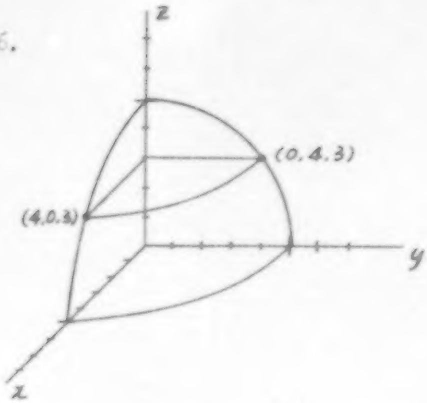
$$-9x^2 + 4y^2 + 4z^2 = 0 .$$

5.



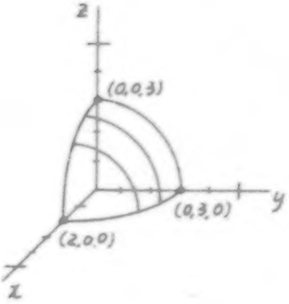
$$x^2 + y^2 + z^2 = 25 .$$

6.



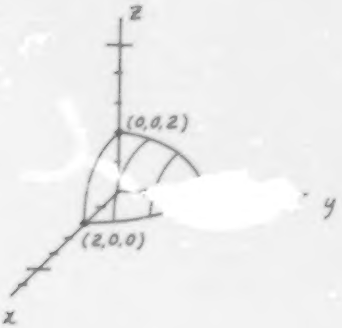
$x^2 + y^2 + z^2 = 25$.

7.


z
(0,0,3)
(0,3,0)
y
(2,0,0)
x

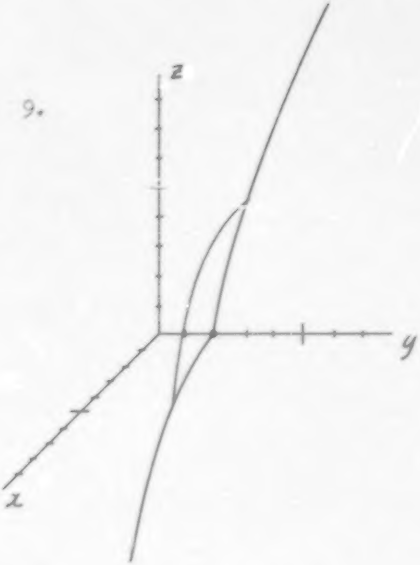
$9x^2 + 4y^2 + 4z^2 = 36$.

8.



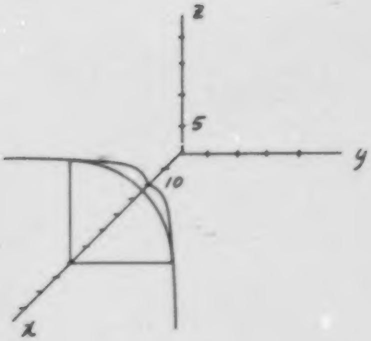
$$9x^2 + 4y^2 + 9z^2 = 36 .$$

9.



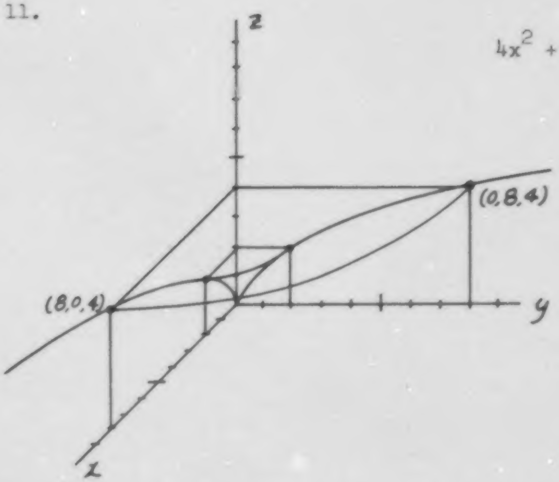
$$x^2 - 4y^2 + z^2 + 16 = 0 .$$

10.



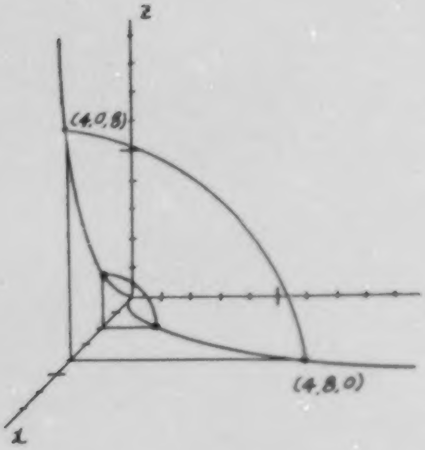
$-x^2 + 4y^2 + 4z^2 + 100 = 0$.

11.



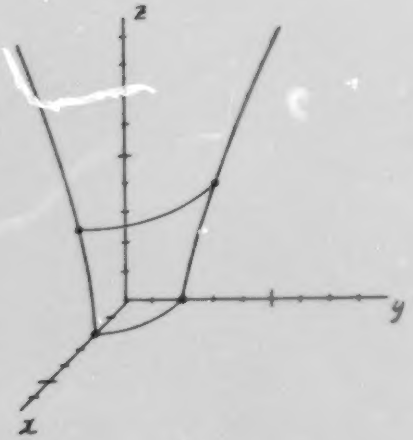
$4x^2 + 4y^2 - z^4 = 0$.

12.



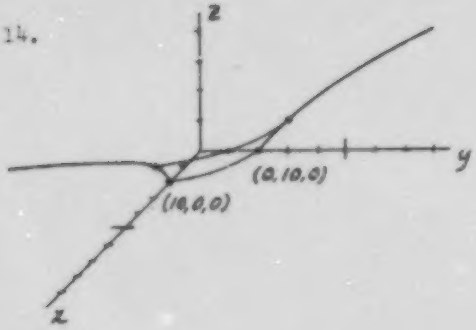
$-x^4 + 4y^2 + 4z^2 = 0$.

13.



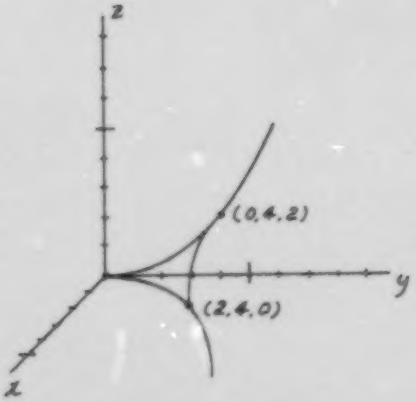
$4x^2 + 4y^2 - z^2 = 16$.

14.


z
y
x
(0,10,0)
(10,0,0)

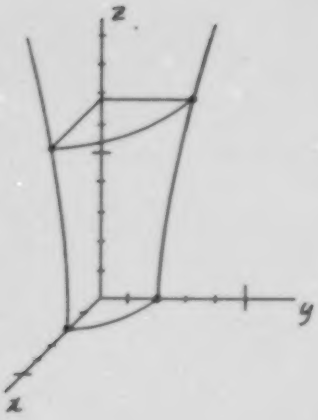
$x^2 + y^2 - 4z^2 = 100$.

15.



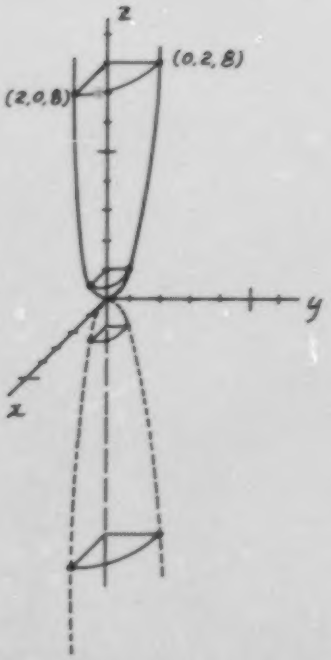
$64x^2 - y^4 + 64z^2 = 0$.

16.



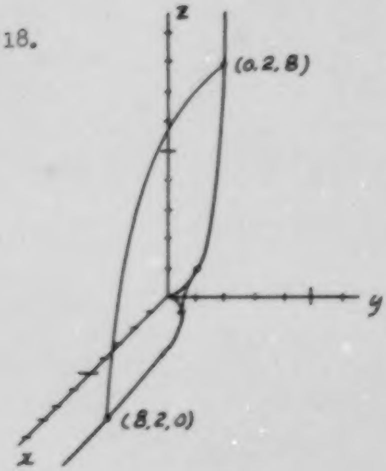
$9x^2 + 9y^2 - z^2 = 36 .$

17.



$(x^2 + y^2)^3 - z^2 = 0 .$

18.



$x^2 - y^6 + z^2 = 0$.

19. Since this is a surface of revolution about the y-axis, any section parallel to the xz-plane will be a circle of radius k with equation $x^2 + z^2 = k^2$. The number k is the ordinate, z , of any point on the curve $f(y,z) = 0$ in the yz-plane; hence, since $k = \sqrt{x^2 + z^2}$, the equation of the surface is $f(y, \sqrt{x^2 + z^2}) = 0$.

372 Projecting cylinders, although time consuming to draw, can be very helpful in locating a space intersection. Look, for example, at Number 48(a) of the Review Exercises. We have the intersection of a spheroid and a hyperboloid; this is extremely difficult to visualize. But when we employ projecting cylinders, we see that the curve lies in a pair of planes through the y-axis and that its projection on the horizontal xy-plane is a circle. This is much easier to visualize. By the way, even if a student should use a different pair of projecting cylinders than we have used, he will obtain the same intersection.

Exercises 9-7

1. (a) $x^2 + z^2 = 12$, $y = -2$. A circle; radius, $\sqrt{12}$; center on y-axis; parallel to and 2 units left of the xz-plane.

(b) $y^2 + z^2 = -5$, $x = 3$. No intersection. (This first equation represents an imaginary cylinder.)

(c) $x^2 + y^2 = 4$, $z = 0$. A circle; radius, 2; center at origin; in the xy-plane.

(d) $x^2 + y^2 = 4$, $z = 0$. Same locus as part (c).

(e) $x^2 + z^2 = 5$, $y = 5$. A circle; radius, $\sqrt{5}$; center on y-axis; parallel to and 5 units right of the xz-plane.

(f) $x^2 = 25$, $z = 0$. A pair of lines; parallel to the y-axis; 5 units on opposite sides of the y-axis in the xy-plane.

(g) $x^2 = 25$, $x - y = 0$. A pair of lines; parallel to the z-axis; 5 units on opposite sides of the z-axis in the plane which bisects the first octant.

(h) $x^2 + 8y^2 = 16$, $z = 1$. An ellipse; center on z-axis; parallel to and 1 unit above the xy-plane.

(i) $3y^2 - 4z^2 = 12$, $x = 0$. A hyperbola; center at the origin; in the yz-plane.

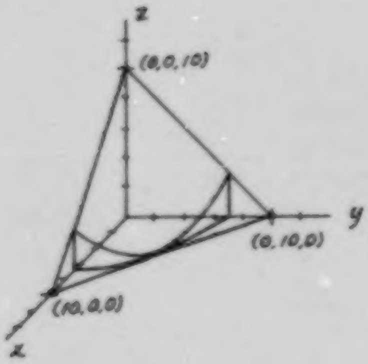
(j) $2y^2 + 8z^2 = 8$, $x = 0$. An ellipse; center at the origin; in the yz-plane.

(k) $x^2 + 8z^2 = 0$, $y = 2$. The point (0,2,0). (This first equation represents a degenerate elliptical cylinder--the y-axis.)

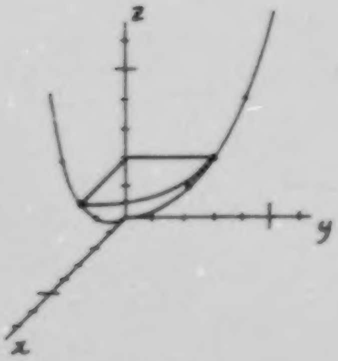
(ℓ) $x^2 + y^2 = 2$,
$z = 1$.

A circle; radius, $\sqrt{2}$; center on the z-axis; parallel to and one unit above the xy-plane. (Hint: subtract the second equation from the first and substitute for z .)

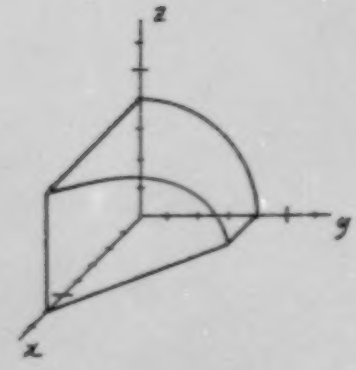
2. (a)


z
(0,0,10)
y
(0,10,0)
(10,0,0)
x

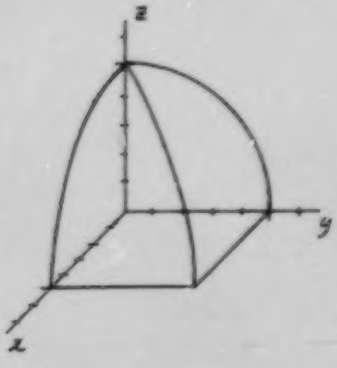
(c)



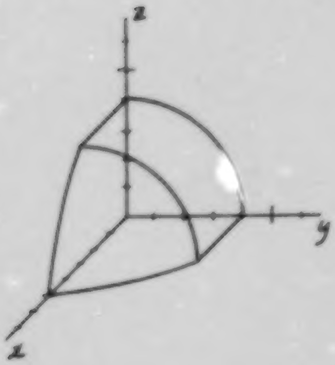
(b)



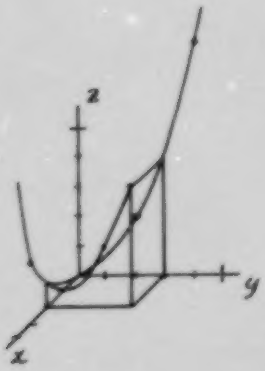
(d)



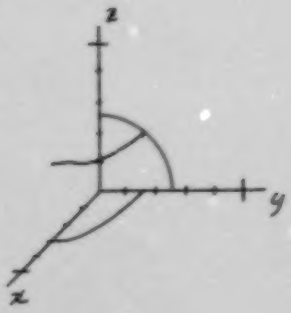
(e)



(f)



3.

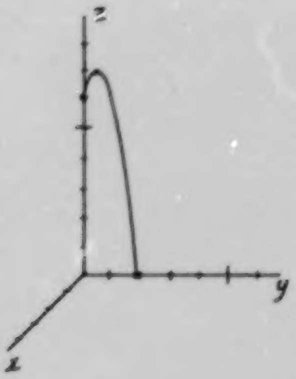


$y^2 + z^2 = 6$;

$-x^2 + 3z^2 = 9$;

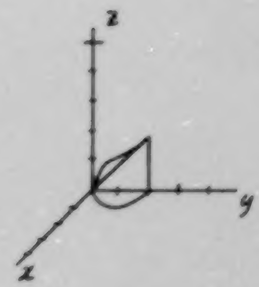
$x^2 + 3y^2 = 9$.

4.



The point is 14 units above the xy-plane. By eliminating x, the equation of the projecting cylinder with elements parallel to the x-axis is $z - 14 = -2(y - 1)^2$. The yz-trace of this parabolic cylinder is a parabola, which shows the projection of the highest point of the space curve, $(\frac{1}{2}, 1, 14)$. Interested students may wish to find this point by observing the projection in the xz-plane.

5.



$z^2 = x^2 + y^2$ represents one projecting cylinder, and eliminating x from the other equation gives

$z^2 - 2y = 0$, which represents a projecting (parabolic) cylinder with elements parallel to the x-axis. The resulting space curve together with the xy- and yz-traces completes the outline of the figure.

377 Since cylindrical and spherical coordinates make use of polar forms, the remarks made previously about the ambiguity of this type of representation apply here as well.

There is nothing (except custom) to prevent us from applying the polar designations to one of the other coordinate planes. Thus, in Figure 9-23, we might have designated the point as $P = (r,y,\theta)$ or $P = (x,r,\theta)$. We chose the form which is in common use.

Exercises 9-8

1. $r = \rho \sin \phi$, $\quad$ $\rho^2 = r^2 + z^2$,

   $\theta = \theta$, $\quad$ or $\quad$ $\theta = \theta$

   $z = \rho \cos \phi$. $\quad$ $\tan \phi = \frac{r}{z}$

   These may be obtained by equating the cylindrical and spherical forms in terms of rectangular coordinates. For example,

$$r \cos \theta = x = \rho \sin \phi \cos \theta \text{ ,}$$

$$r = \rho \sin \phi \text{ .}$$

$$\tan \theta = \frac{y}{x} = \frac{\rho \sin \phi \sin \theta}{\rho \sin \phi \cos \theta} \text{ ,}$$

$$\tan \theta = \frac{\sin \theta}{\cos \theta} \text{ , (an identity); therefore,}$$

$$\theta = \theta \text{ .}$$

2. (a) $(\sqrt{2}, \sqrt{6}, 2\sqrt{2})$; $(2\sqrt{2}, \frac{\pi}{3}, 2\sqrt{2})$.

   (b) $(\frac{3}{2}\sqrt{3}, 0, \frac{3}{2})$; $(\frac{3}{2}\sqrt{3}, 0, \frac{3}{2})$.

   (c) $(0,2,0)$; $(2, \frac{\pi}{2}, 0)$.

   (d) $(0.239, 3.354, 2.160)$; $(3.364, \frac{3}{2}, 2.160)$.

3. (a) $(\sqrt{3}, 1, 3)$; $(\sqrt{13}, \frac{\pi}{6}, .59)$.

   (b) $(0,5,0)$; $(5, \frac{\pi}{2}, \frac{\pi}{2})$.

   (c) $(0,0,8)$; $(8, \frac{\pi}{4}, 0)$.

   (d) $(2.160, 3.364, 2)$; $(2\sqrt{5}, 1, 1.11)$.

4. $\rho^2 = x^2 + y^2 + z^2$,

$\tan \theta = \frac{y}{x}$,

$\tan \phi = \frac{\sqrt{x^2 + y^2}}{z}$.

(a) $(\sqrt{13}, .98, 0)$; $(\sqrt{13}, .98, \frac{\pi}{2})$.

(b) $(6, \frac{\pi}{2}, 3)$; $(3\sqrt{5}, \frac{\pi}{2}, 1.11)$.

(c) $(4, \frac{\pi}{6}, 4)$; $(4\sqrt{2}, \frac{\pi}{6}, \frac{\pi}{4})$.

(d) $(\sqrt{17}, .24, 2)$; $(\sqrt{21}, .24, 1.34)$.

5. (a) $r^2 = 25$, or simply $r = 5$; $\rho^2 \sin^2 \phi = 25$, or $\rho \sin \phi = 5$.

(b) $z = 4 \tan \theta$; $\rho \cos \phi \cot \theta = 4$.

(c) $r = 8 \cos \theta$; $\rho \sin \phi = 8 \cos \theta$.

(d) $r^2 = 3z$; $\rho \sin \phi \tan \phi = 3$.

6. (a) $x^2 + y^2 + z^2 = 36$.

(b) $x^2 + y^2 = 36$.

(c) $x^2 + y^2 - (z - 6)^2 = 0$.

(d) $x^2 + y^2 + z^2 = 9$.

7. (a) A cylinder of radius 3 whose axis is the z-axis.

(b) A plane containing the z-axis and bisecting the first octant.

(c) A sphere of radius 2 with center at the origin.

(d) A circular cone whose vertex is at the origin and whose axis is the z-axis.

(e) A plane parallel to and 7 units above the xy-plane.

(f) A plane containing the y-axis and bisecting the first octant.

(g) A circular cone whose vertex is at the origin and whose axis is the z-axis.

(h) A plane parallel to and 2 units in front of the yz-plane.

8. (a) Let the center of the sphere be the origin and the axis of the cylinder have rectangular equations $x = 2$, $y = 0$; then the bounding surfaces are

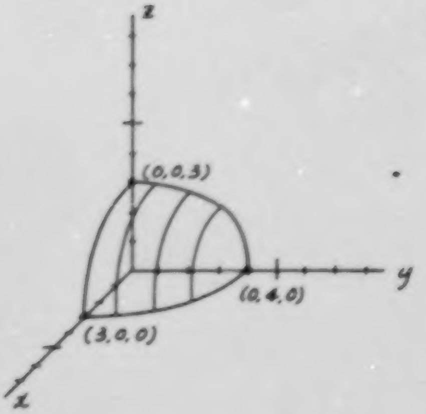
$$x^2 + y^2 + z^2 = 16 ,$$
$$(x - 2)^2 + y^2 = 4 .$$

(b) $r^2 + z^2 = 16$,
$r = 4 \cos \theta$.

(c) $\rho = 4$,
$\rho \sin \phi = 4 \cos \theta$.

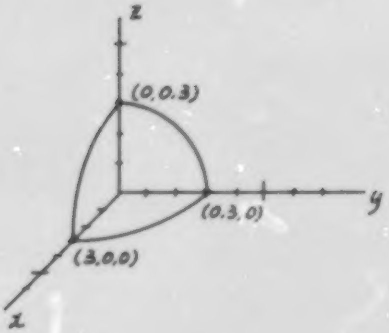
Review Exercises

1.



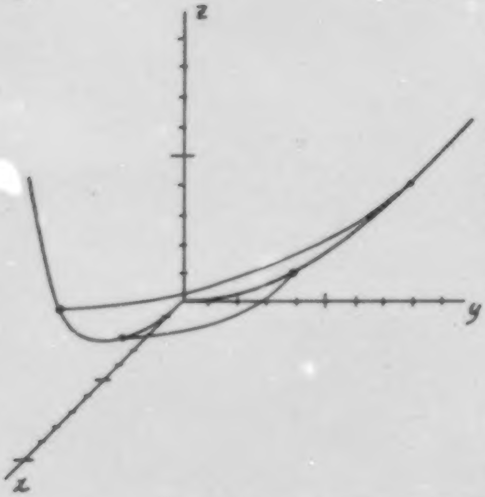
A prolate spheroid.
Sections parallel to the xz-plane are circles. Sections parallel to the other coordinate planes are ellipses.

2.



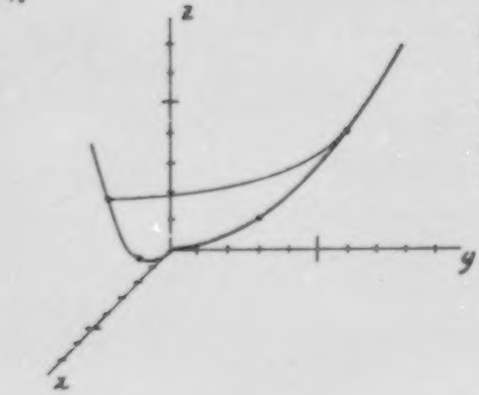
A sphere. Radius: 3 .
All sections are circles.

3.



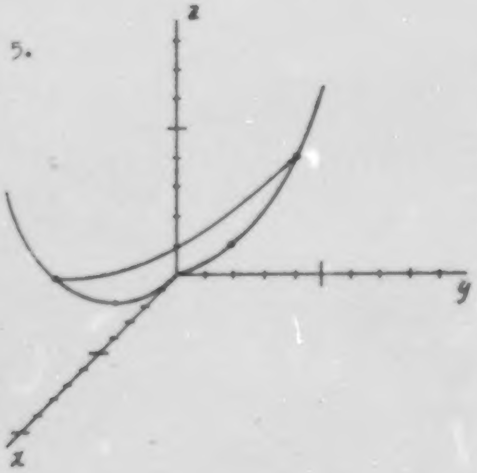
A paraboloid of revolution.
Sections parallel to the xy-plane are circles.
Sections parallel to the other coordinate planes are parabolas.

4.



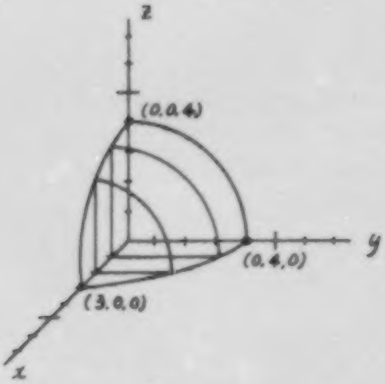
An elliptic paraboloid.
Sections parallel to the xy-plane
are ellipses.
Sections parallel to the other
coordinate planes are parabolas.

5.



An elliptic paraboloid.
Sections parallel to the xy-plane
are ellipses.
Sections parallel to the other
coordinate planes are parabclas.

6.

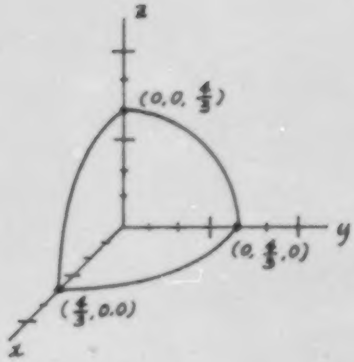

z
(0,0,4)
y
(0,4,0)
(3,0,0)
x

An oblate spheroid.

Sections parallel to the yz-plane are circles.

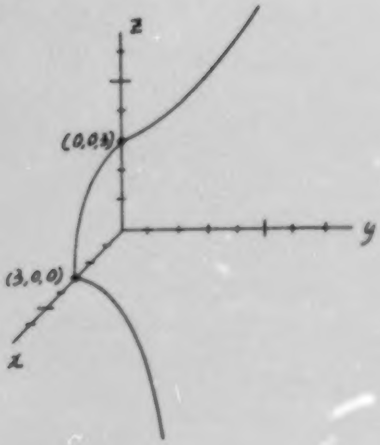
Sections parallel to the other coordinate planes are ellipses.

7.


z
$(0,0,\frac{4}{3})$
y
$(0,\frac{4}{3},0)$
$(\frac{4}{3},0,0)$
x

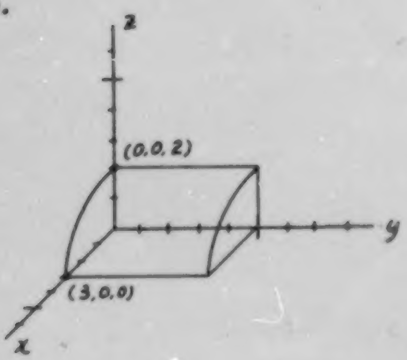
A sphere. Radius: $\frac{4}{3}$.

8.



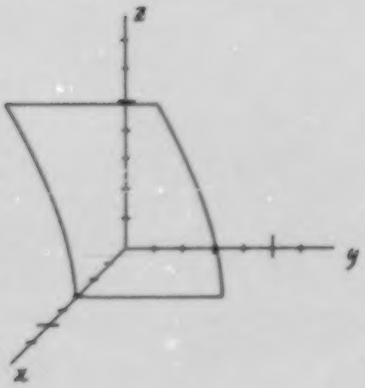
A hyperboloid of revolution (one sheet).
Sections parallel to the xz-plane are circles.
Sections parallel to the other coordinate planes are hyperbolas.

9.



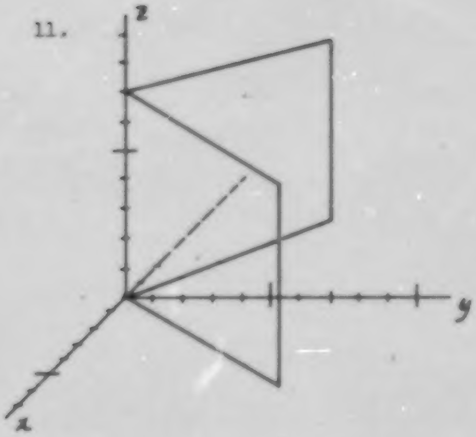
An elliptic cylinder.
Sections parallel to the xz-plane are ellipses.
Sections parallel to the other coordinate planes are parallel lines.

10.



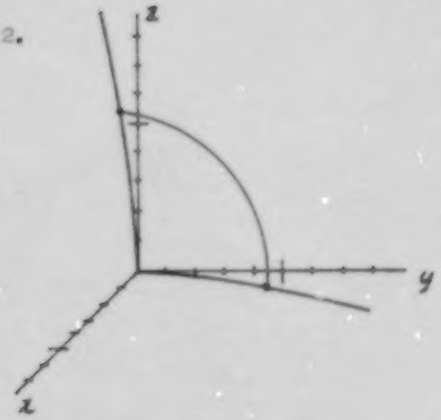
A hyperbolic cylinder (two parts).
Sections parallel to the xz-plane are hyperbolas.
Sections parallel to the other coordinate planes are parallel lines.

11.



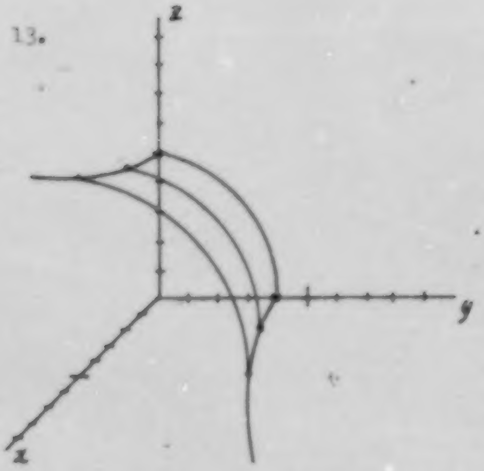
A pair of planes intersecting on the z-axis.

12.



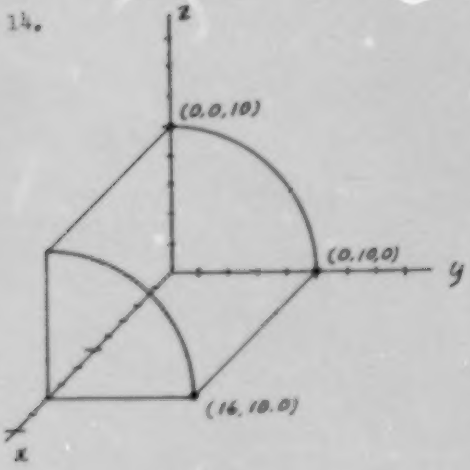
An elliptic paraboloid.
Sections parallel to the yz-plane are ellipses.
Sections parallel to the other coordinate planes are parabolas.

13.



An elliptic hyperboloid (one sheet).
Sections parallel to the yz-plane are ellipses.
Sections parallel to the other coordinate planes are hyperbolas.

14.

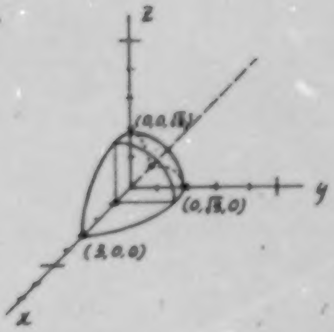

z
(0,0,10)
(0,10,0)
y
(10,10,0)
x

A circular cylinder; axis: x-axis; radius: 10 .

Sections parallel to the yz-plane are circles.

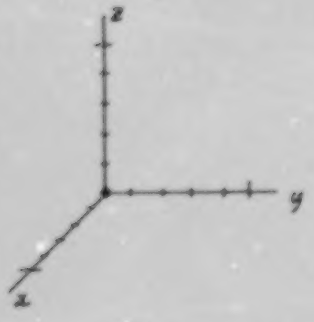
Sections parallel to the other coordinate planes are parallel lines.

15.


z
(0,0,√3)
(0,√3,0)
y
(3,0,0)
x

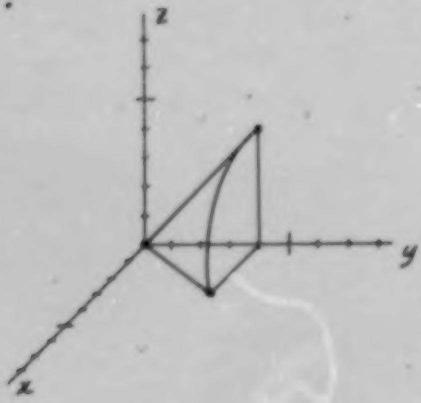
A sphere; center: (1,0,0) ; radius: 2 .

16.



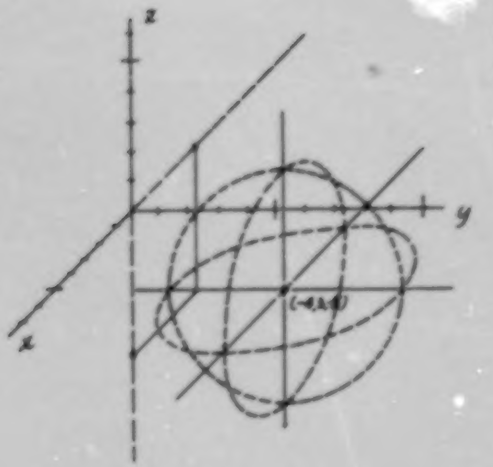
The point (0,0,0) .

17.



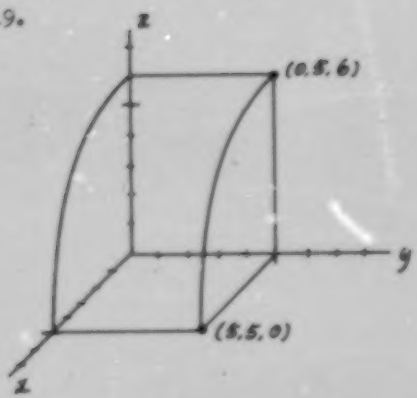
An elliptic cone.
Sections parallel to the xz-plane
are ellipses.
Sections parallel to the other
coordinate planes are hyperbolas.

18.



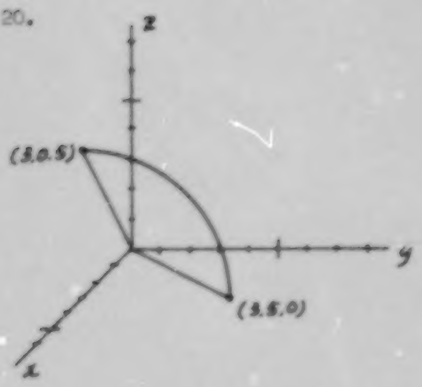
$(x + 4)^2 + (y - 3)^2 + (z + 5)^2 = 16$.
A sphere with radius 4 and center at $(-4,3,-5)$.

19.



An elliptic cylinder.
Sections parallel to the xz-plane are ellipses.
Sections parallel to the other coordinate planes are parallel lines.

20.

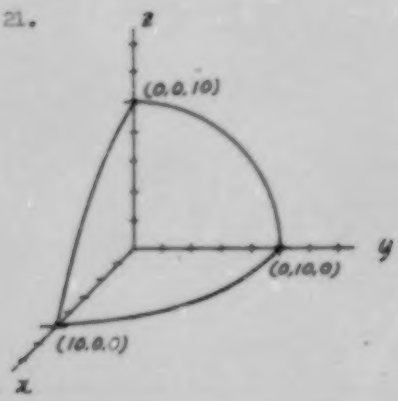


A circular cone.

Sections parallel to the yz-plane are circles.

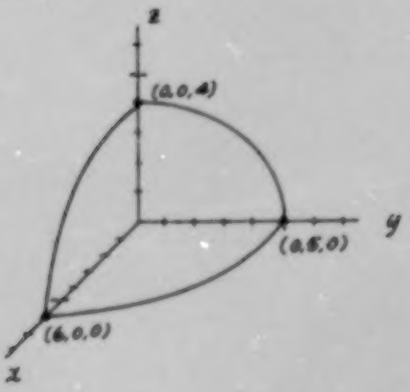
Sections parallel to the other coordinate planes are hyperbolas.

21.


z
y
x
(0,0,10)
(0,10,0)
(10,0,0)

$x^2 + y^2 + z^2 = 100$.

22.

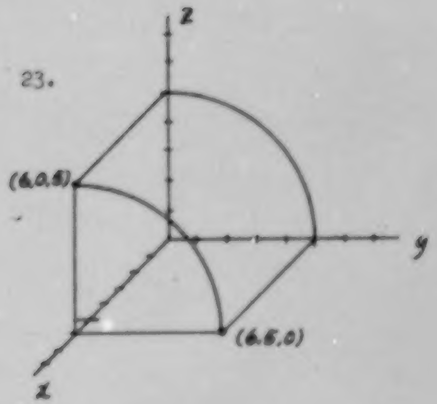

z
(0,0,4)
y
(0,5,0)
(6,0,0)
x

Assume that the ellipsoid has its center at the origin.

$$\frac{x^2}{36} + \frac{y^2}{25} + \frac{z^2}{16} = 1 ,$$

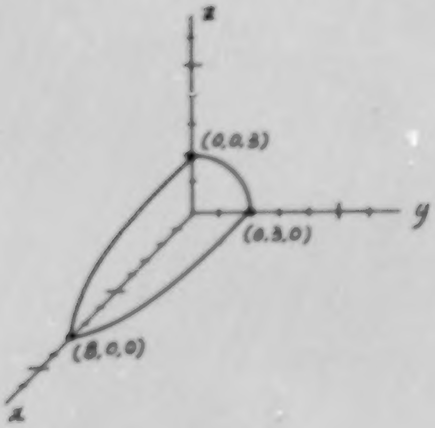
or $100x^2 + 144y^2 + 225z^2 = 3600$.

23.



$y^2 + z^2 = 25$.

24.

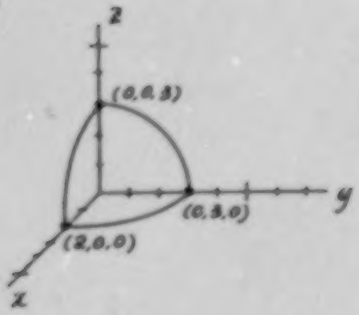

z
(0,0,2)
(0,2,0)
y
(8,0,0)
x

Assume the spheroid located as shown.

$$\frac{x^2}{64} + \frac{y^2}{4} + \frac{z^2}{4} = 1 \text{ , or}$$

$$x^2 + 16y^2 + 16z^2 = 64 \text{ .}$$

25.

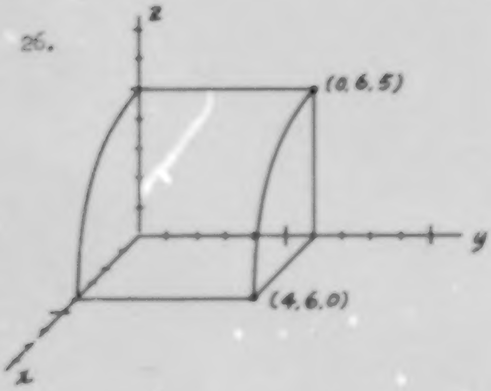

z
(0,0,3)
y
(0,3,0)
(2,0,0)
x

Assume the spheroid located as shown.

$$\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{9} = 1 \text{ , or}$$

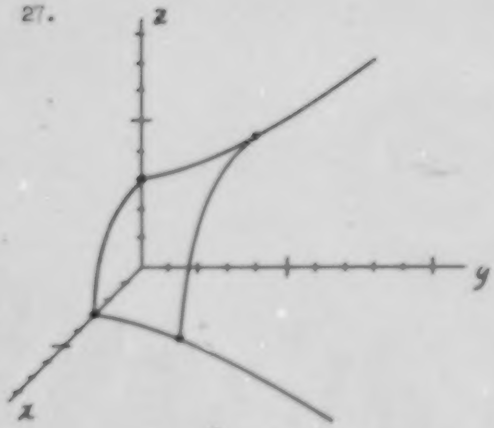
$$9x^2 + 4y^2 + 4z^2 = 36 \text{ .}$$

26.



$25x^2 + 16z^2 = 400$.

27.

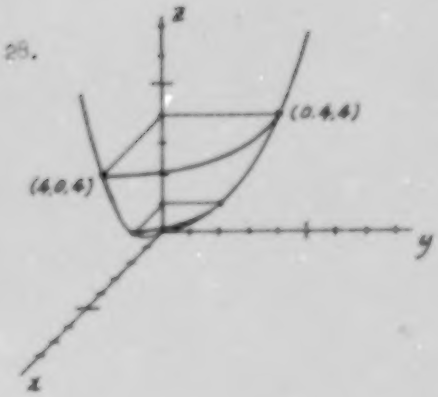


A hyperboloid of one sheet.

$$\frac{x^2}{9} - \frac{y^2}{16} + \frac{z^2}{9} = 1 \text{ , or}$$

$$16x^2 - 9y^2 + 16z^2 = 144 \text{ .}$$

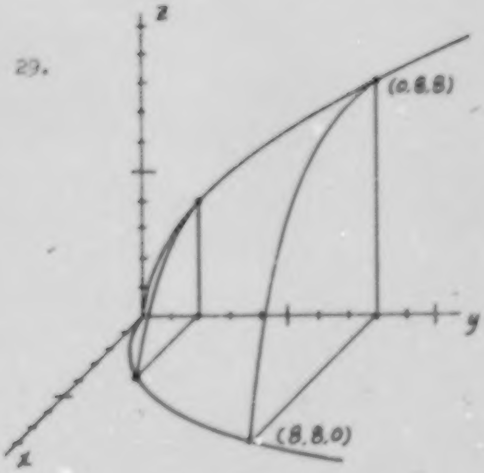
28.



A paraboloid.

$x^2 + y^2 = 4z$.

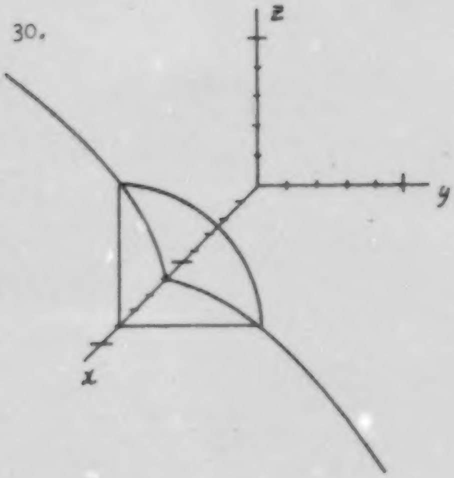
29.



A paraboloid.

$x^2 + z^2 = 8y$.

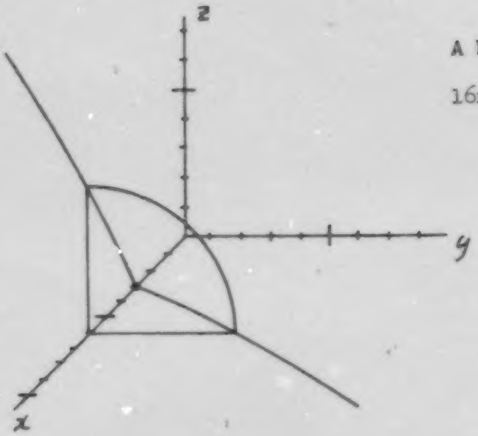
30.



A hyperboloid of two sheets.

$25x^2 - 36y^2 - 36z^2 = 900$.

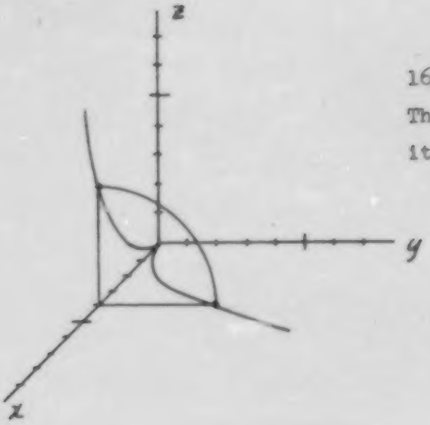
31.



A hyperboloid of two sheets.

$16x^2 - 9y^2 - 9z^2 = 144$.

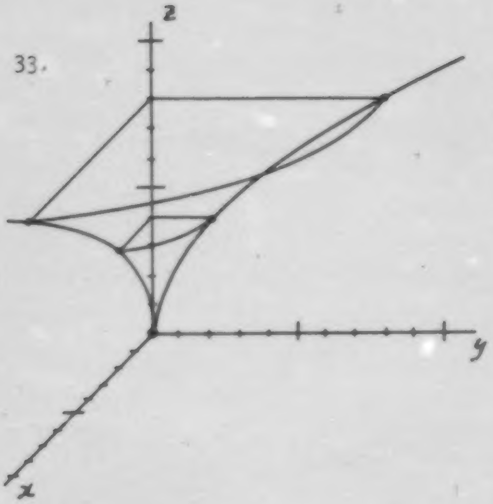
32.



$16y^2 + 16z^2 = x^4$.

This is not a quadric surface; it resembles Figure 9-19.

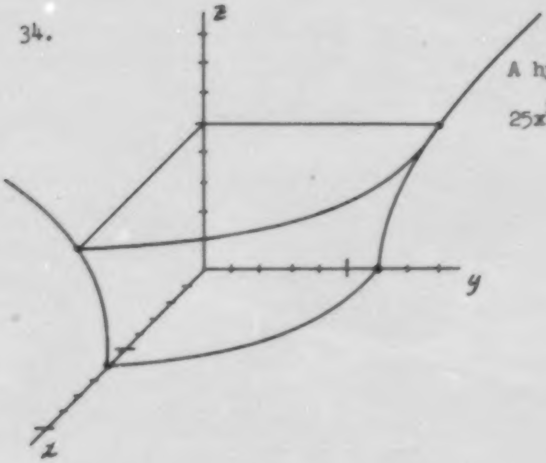
33.



$64x^2 + 64y^2 = z^4$.

Compare with Figure 9-19.

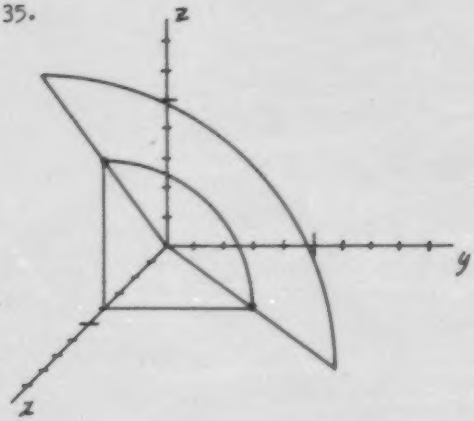
34.



A hyperboloid of one sheet

$25x^2 + 25y^2 - 36z^2 = 900$.

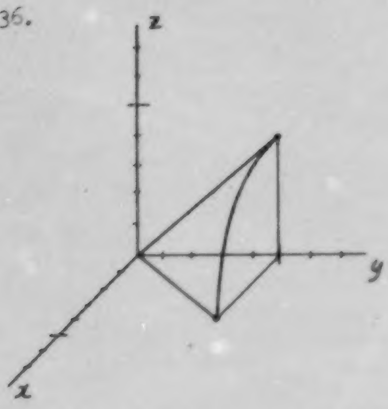
35.



A circular cone.

$-25x^2 + 16y^2 + 16z^2 = 0$

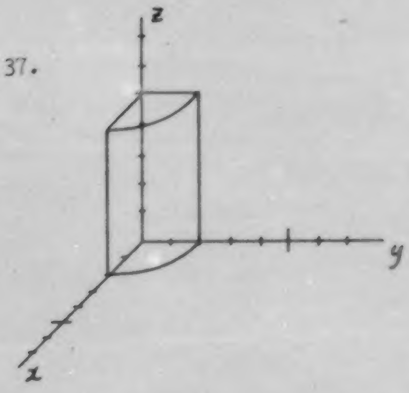
36.



A circular cone.

$25x^2 - 16y^2 + 25z^2 = 0$.

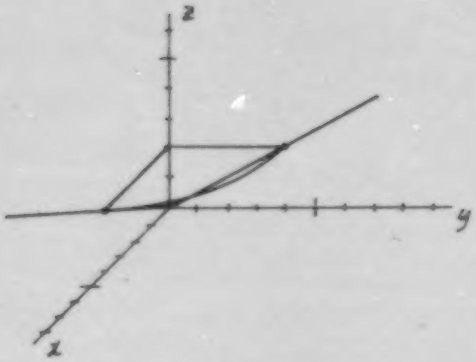
37.



A circular cylinder.

$x^2 + y^2 = 4$.

38.



A circular cone.

$x^2 + y^2 - 4z^2 = 0$.

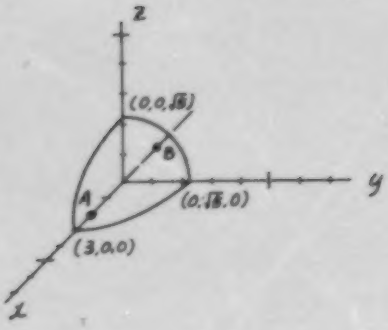
39. $x^2 + y^2 + z^2 - 6x + 4y - 2z - 86 = 0$.

40. $x^2 + z^2 = 25$.

41. $y^2 + z^2 = 12x - 36$.

42. (a) Ellipsoid.
    (b) A point.
    (c) No locus.
    (d) Elliptic hyperboloid of one sheet.
    (e) Elliptic cone.
    (f) Elliptic hyperboloid of two sheets.
    (g) Elliptic hyperboloid of two sheets.
    (h) Elliptic cone.
    (i) Elliptic hyperboloid of one sheet.

43.



Let the points be $A = (2,0,0)$ and $B = (-2,0,0)$. The equation is

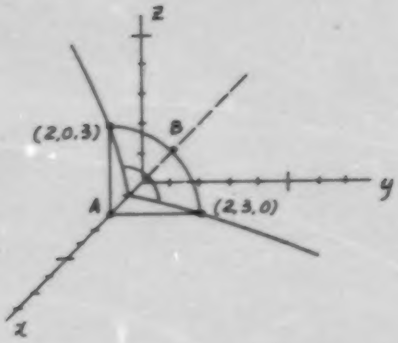
$$\sqrt{(x-2)^2+y^2+z^2} + \sqrt{(x+2)^2+y^2+z^2} = 6.$$

This simplifies to

$$5x^2 + 9y^2 + 9z^2 = 45 ,$$

an equation of a prolate spheroid.

44.



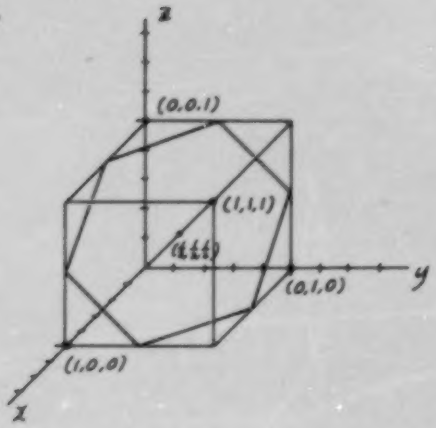
The equation is

$$\sqrt{(x-2)^2+y^2+z^2} - \sqrt{(x+2)^2+y^2+z^2} = 2,$$

or $3x^2 - y^2 - z^2 = 3$,
an equation of a hyperboloid of two sheets.

45. Since the blades of the sharpener generate a circular cone and the sides of the pencil are portions of planes parallel to the axis of the cone, the intersection will be, under ideal conditions, portions of six congruent hyperbolas.

46.



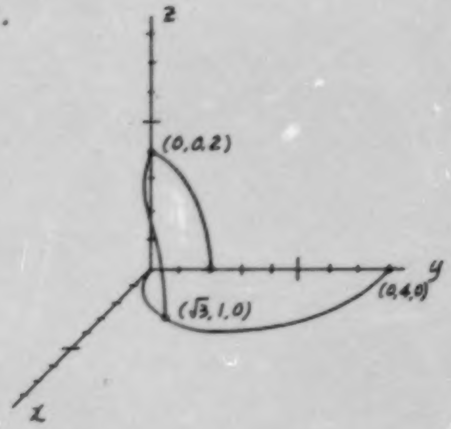
Since the plane contains the point $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, the equation of the plane in normal form is

$$\frac{1}{2}x + \frac{1}{2}y + \frac{1}{2}z = \frac{3}{4}\text{, or}$$

$$x + y + z = \frac{3}{2}.$$

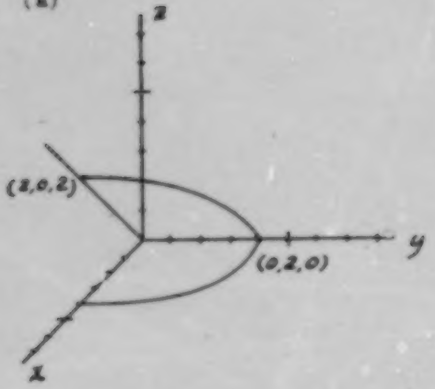
The intersection with the cube is a regular hexagon with sides of length $\frac{1}{2}\sqrt{2}$.

47.



The second equation represents a circular cylinder whose xy-trace is shown. Subtracting the second equation from the first gives $z^2 + 4y = 4$; this equation represents a parabolic cylinder whose yz-trace is shown. The space curve is the intersection of the sphere and cylinder.

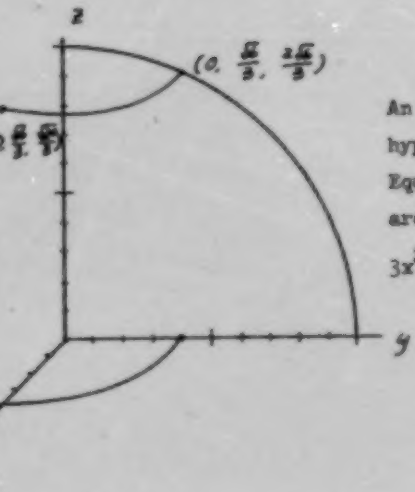
48. (a)



An oblate spheroid and an elliptic hyperboloid of one sheet. Two projecting cylinders have equations

$x^2 - z^2 = 0$, $x^2 + y^2 = 4$.

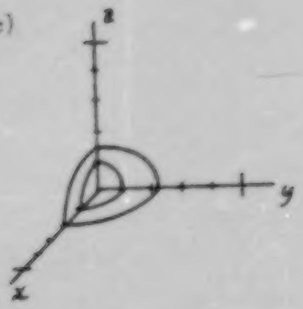
(b)


z
$(0, \frac{\sqrt{6}}{3}, \frac{2\sqrt{6}}{3})$
y
x

An oblate spheroid and an elliptic hyperboloid of two sheets. Equations of two projecting cylinders are

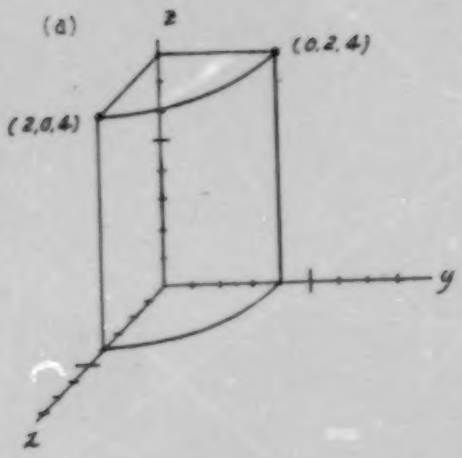
$3x^2 + 3y^2 = 2$, $3y^2 + 3z^2 = 10$.

(c)



A sphere and an oblate spheroid. These surfaces do not intersect. The sphere lies entirely inside the spheroid.

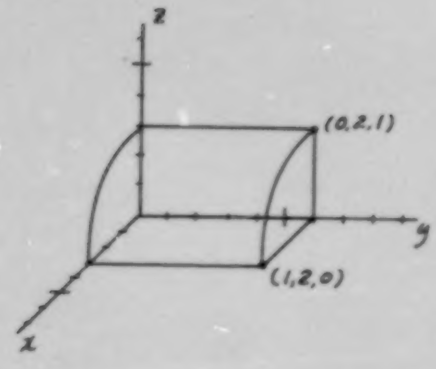
(d)



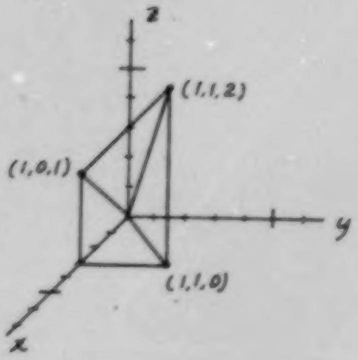
A paraboloid of revolution and a circular cylinder. Equations of two projecting cylinders are

$x^2 + y^2 = 4$, $z = 4$.

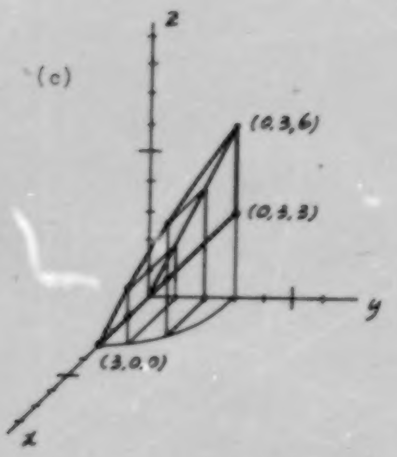
49. (a)



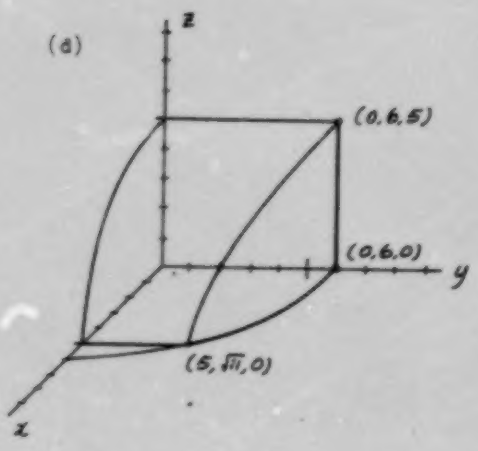
(b)


z
(1,1,2)
(1,0,1)
y
(1,1,0)
x

(c)


z
(0,3,6)
(0,3,3)
y
(3,0,0)
x

(d)


z
(0,6,5)
(0,6,0)
y
(5,√11,0)
x

50. (a) $z = 5$; $\rho \cos \phi = 5$.

(b) $r = 4 \cos \theta$; $\rho \sin \phi = 4 \cos \theta$.

(c) $x^2 + y^2 = 49$; $\rho \sin \phi = 7$.

(d) $r^2 + z^2 = 25$; $\rho = 5$.

(e) $x^2 + y^2 + z^2 = 9$; $\rho = 3$.

(f) $z = 6$; $z = 6$.

(g) $r^2(\cos^2 \theta - \sin^2 \theta) = 16$; $\rho^2 \sin^2 \phi (\cos^2 \theta - \sin^2 \theta) = 16$.

(h) $x^2 + y^2 = 2x$; $\rho \sin \phi = 2 \cos \theta$.

(i) $x^2 + y^2 = 2z$; $r^2 = 2z$.

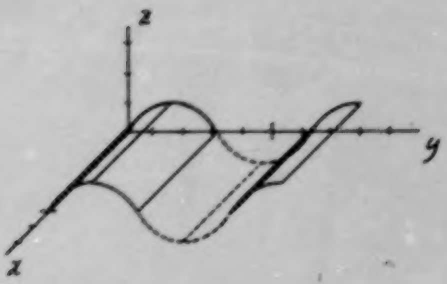
(j) $x^2 + y^2 = 9$; $r = 3$.

(k) $r = 8$; $\rho \sin \phi = 8$.

(ℓ) $x = yz$; $z = \cot \theta$.

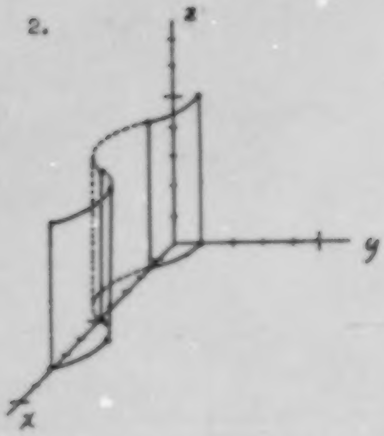
Challenge Problems

1.



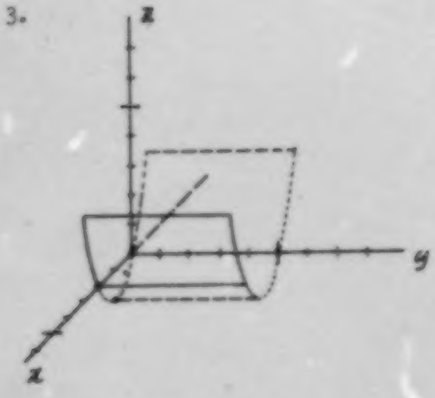
A cylinder with elements parallel to the x-axis and whose directrix is a sine curve.

2.



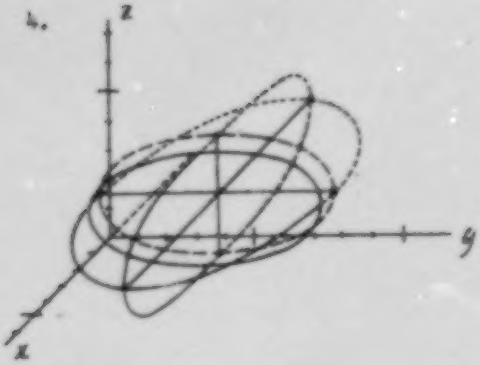
A cylinder with elements parallel to the z-axis and whose directrix is a cosine curve.

3.



A parabolic cylinder with elements parallel to the y-axis .

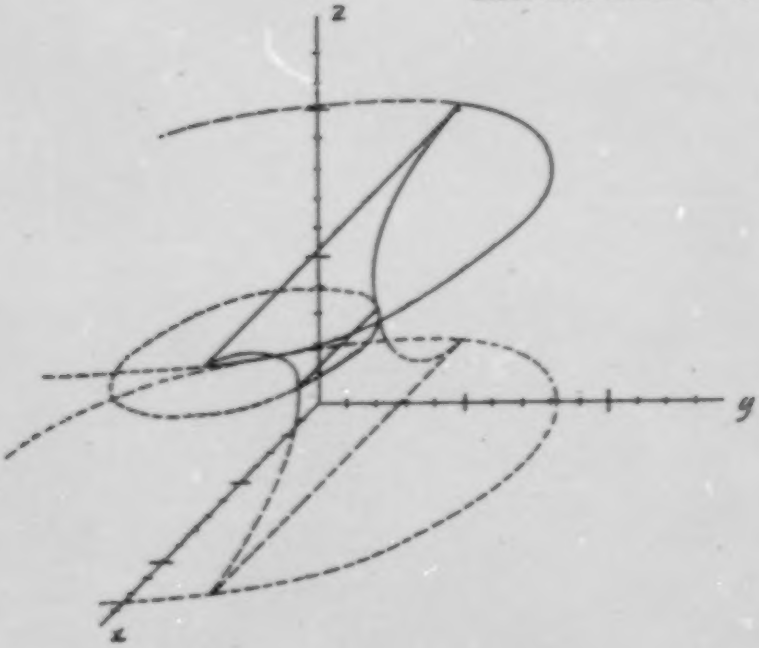
4.



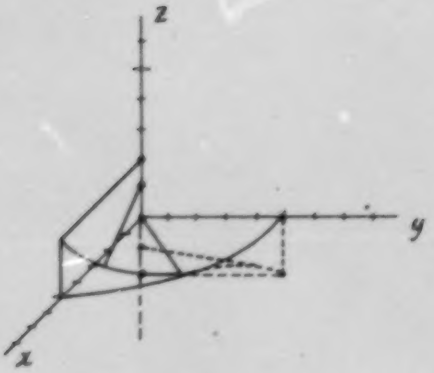
An ellipsoid with center at (-1,3,1). The axis parallel to the x-axis has length 12; the axis parallel to the y-axis has length 8; the axis parallel to the z-axis has length 4.

5.

A hyperboloid of revolution of one sheet with center at (-1,-3,1).



6.



This curious surface is a type of ruled surface. It is entirely contained between the planes $z = 1$ and $z = -1$. It can be visualized as being generated by a line (minus the point on the z-axis) which is always perpendicular to the z-axis. Starting in the plane $z = 1$ and parallel to the x-axis, the line rotates as it drops until it is parallel to the y-axis in the plane $z = -1$. The line continues to rotate as it rises to its original position, thus completing the surface.

Teachers' Commentary

Chapter 10

GEOMETRIC TRANSFORMATION

10-1. Why Study Geometric Transformations?

Most of this chapter is an extension of Chapters 5-7 on curve tracing and conics. Although the principles presented in Sections 10-2 and 10-5 are applicable to all curves, we emphasized the straight line and the conics because of their importance and because the students are more familiar with them.

The treatment of geometric transformations in this text differs from most other texts in that we look upon geometric transformations from *two* points of view. We first move the axes, keeping the figure fixed; then we move the figure, keeping the axes fixed. We feel that the student should become acquainted with both types. The point transformation has much wider application than the transformation of axes and Section 6 was included to show its possibilities.

Groups of transformations are discussed in a supplementary chapter. You will recall that in 1872, Professor Felix Klein (1849-1925) presented his famous "Erlanger Program" in which he classified all geometrics on the basis of those properties invariant under groups of transformations. Mention is made in this chapter of the set of rigid motions which characterize Euclidean Geometry without designating them as a group. You will find a good treatment of this subject in Courant and Robbins' book entitled *What is Mathematics?*

10-2. Translations.

Sufficient motivation may be provided for this section by requiring the students to graph each part of the equations listed in the first section.

You will note that there are two forms given for the equations of translation. The form $\left\{\begin{array}{l} x' = x + h \\ y' = y + k \end{array}\right.$ is more useful for the translation of axes because more applications are similar to those presented in Examples 3 and 4.

NOTE: Almost all solutions for this chapter are presented without graphs since they are so familiar to you.

Exercises 10-2

1. $\left\{\begin{array}{l} x' = x + 3 \\ y' = y - 4 \end{array}\right.$,

   $O' = (-3,4)$

2. The equations of translation are $\left\{\begin{array}{l} x = x' + 3 \\ y = y' - 2 \end{array}\right.$.

   The new equation is:

   $$2(x' + 3)^2 - (y' - 2)^2 - 12(x' + 3) - 4(y' - 2) + 12 = 0 ,$$

   which simplifies to $2x'^2 - y'^2 = 2$.

3. (a) $(x' - 4)^2 + (y' - 6)^2 = r^2$. This is a circle with the same radius, and center at $(4,6)$.

   (b) $\dfrac{(x' - 4)^2}{a^2} - \dfrac{(y' - 6)^2}{b^2} = 1$.

   This is a congruent hyperbola with its center at $(4,6)$ and with its axes parallel to the x- and y-axes.

   Neither of the curves undergo a change. They merely have a new equation relative to the new axes.

4. The equations of translation are $\left\{\begin{array}{l} x' = x + 4 \\ y' = y + 2 \end{array}\right.$. The new coordinates of the vertices of the triangle are $A = (5,2)$, $B = (9,0)$, and $C = (7,6)$ with reference to the new origin. Two suggested methods are:

   (a) Application of the Pythagorean Theorem.

   (b) Proof that the product of the slopes of $\overline{AB}$ and $\overline{AC}$ equals -1 .

5. By completing the square, the equation of the hyperbola becomes

(1) $$(x + 5)^2 - (y - 2)^2 = 16 .$$

Substituting x' for $(x + 5)$ and y' for $(y - 2)$ into (1), we have

(2) $$x'^2 - y'^2 = 16 .$$

Equation (2) represents the same hyperbola with reference to the new axes with origin $O' = (-5,2)$.

To graph, translate the origin to O'. Draw the x'- and y'-axes through O'. Then draw the graph of Equation (2) with respect to the x'- and y'-axes.

6. The students may select any three points. We choose $A = (5,0)$, $B = (3,4)$, and $C = (0,5)$. After translation, the coordinates of $A = (4,1)$, $B = (2,5)$, and $C = (-1,6)$ with respect to the new origin. The transformed equation is $(x' + 1)^2 + (y' - 1)^2 = 25$. The new coordinates satisfy this equation.

7. After the first translation, L has the equation $3x' - 2y' + 11 = 0$ with respect to the x'- and y'-axes. After the second translation, L has the equation $3x'' - 2y'' + 13 = 0$ with respect to the x''- and y''-axes. The transformation $\begin{cases} x = x'' + 7 \\ y = y'' + 7 \end{cases}$ would have the same effect as the two successive ones.

There would be no difference in the final result if these translations were commuted. (This is true of all translations.)

8. (a) Completing the square, the equation becomes $(y - 3)^2 = 12(x + 1)$. The equations of translation are therefore:

$$\begin{cases} x' = x + 1 \\ y' = y - 3 \end{cases}$$

The parabola now has the equation $y'^2 = 12x'$ with respect to the new origin at $O' = (-1,3)$. To graph, draw the x'- and y'-axes through O' and sketch the new equation with respect to those axes.

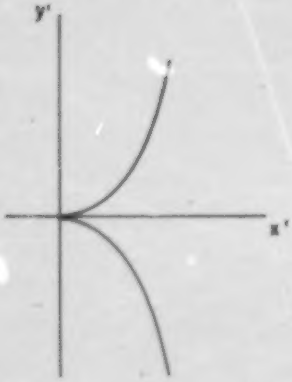
The solution of the other parts is similar to part (a). The new equations and origin are:

(b) $\frac{x'^2}{4} + \frac{y'^2}{3} = 1$; $O' = (1,-1)$.

(c) $x'^2 = \frac{3}{2}y'$; $O' = (-\frac{3}{2}, \frac{5}{2})$.

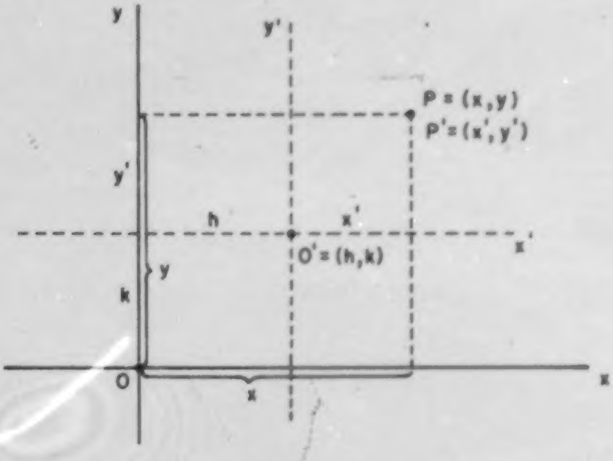
(d) $x'y' = 12$; $O' = (-3,4)$.

(e) $y'^2 = x'^3$; $O' = (-2,-2)$.



The graph of (e) looks like the figure to the right. It is called a semi-cubic parabola. There is no asymptote.

9.



From the figure, $\begin{cases} x = x' + h \\ y = y' + k \end{cases}$ or $\begin{cases} x' = x - h \\ y' = y - k \end{cases}$.

10-3. Rotation of Axes. Rectangular Coordinates.

Motivation for this section, as for Section 10-2, could be provided by asking the students to graph the pair of equations

$$x^2 + 2\sqrt{3}\,xy - y^2 = 8 \quad \text{and} \quad x'^2 - y'^2 = 4 .$$

Then point out to them or have them discover that the graphs are identical except for position.

You will note that we present two forms for the equations of rotation. The form chosen for use in solving a given problem depends upon the nature and form of the problem. The four examples presented in the text should clarify this point.

Your better students should be encouraged to study the Supplement to Chapter 7 where the topic is discussed in detail. Among other things, the student will learn how to determine the angle of rotation in order to arrive at a new set of axes and an equation containing no xy-term.

402 The "digression" on this page, which discusses the merits of one form of an equation over another for the same curve when both are simple, has an ulterior motive. That purpose is to indicate several scientific areas in which the equilateral hyperbola, in the form $xy = k$, is studied. As a rule, students are more acquainted with the other conics.

403 We recommend that the details of this rotation be carried out in class by the instructor. The students may then carry out the details of rotating the axes through an angle of measure α, and arrive at the equation of the circle

$$x'^2 + y'^2 + Dx' + Ey' + F = 0$$

with respect to the new $x'y'$-axes. No $x'y'$-term should appear. A complete discussion of the general equation of the second degree is found in the Supplementary Chapter for Chapter 7.

Because of the nature of polar coordinates, the rotation of the polar axis leads to a very simple result. Once again we have restricted ourselves to the conics. If time permits, you may like to discuss the rose curves, lemniscates, spirals, and other curves.

Exercises 10-3

1. Since $\alpha = 150^\circ$, $\sin \alpha = \frac{1}{2}$ and $\cos \alpha = -\frac{\sqrt{3}}{2}$. The equations of rotation are $\begin{cases} x' = \frac{1}{2}(-\sqrt{3}x + y) \\ y' = \frac{1}{2}(-x - \sqrt{3}y) \end{cases}$. The new coordinates of the vertices of the triangle are $A = (-\frac{\sqrt{3}}{2}, -\frac{1}{2})$, $B = \left(\frac{-5\sqrt{3} - 2}{2}, \frac{-5 + 2\sqrt{3}}{2}\right)$, and $C = \left(\frac{-3\sqrt{3} + 4}{2}, \frac{-3 - 4\sqrt{3}}{2}\right)$. Using the original coordinates, we

have $d(A,B) = \sqrt{20}$, $d(A,C) = \sqrt{20}$. $\therefore$ Area of $\triangle ABC = 10$. Using the new coordinates, we have

$$d(A,B) = \sqrt{\left(\frac{-5\sqrt{3} - 2}{2} + \frac{\sqrt{3}}{2}\right)^2 + \left(\frac{-5 + 2\sqrt{3}}{2} + \frac{1}{2}\right)^2}$$

$$= \sqrt{(-2\sqrt{3} - 1)^2 + (-2 + \sqrt{3})^2} = \sqrt{20} .$$

Similarly $d(A,C) = \sqrt{20}$, and the area of $\triangle ABC = 10$.

2. Since $\alpha = -30^\circ$, $\sin \alpha = -\frac{1}{2}$ and $\cos \alpha = \frac{\sqrt{3}}{2}$. The equations of rotation are therefore:

$$\begin{cases} x = \frac{1}{2}(\sqrt{3}\, x' + y') \\ y = \frac{1}{2}(-x' + \sqrt{3}\, y') \end{cases} .$$

The equation of the line with respect to the new axes is:

$$\frac{3}{2}(\sqrt{3}\, x' + y') + 1(-x' + \sqrt{3}\, y') - 8 = 0 ,$$

which simplifies to $(3\sqrt{3} - 2)x' + (2\sqrt{3} + 3)y' - 16 = 0$.

The slope is $-\dfrac{3\sqrt{3} - 2}{2\sqrt{3} + 3} = \dfrac{13\sqrt{3} - 24}{3}$.

3. $$\begin{cases} x' \cos \alpha - y' \sin \alpha = x \\ x' \sin \alpha + y' \cos \alpha = y \end{cases}$$

$$\therefore \quad x' \cos^2 \alpha - y' \sin \alpha \cos \alpha = x \cos \alpha$$

$$x' \sin^2 \alpha + y' \sin \alpha \cos \alpha = y \sin \alpha$$

Adding corresponding members, we have

$$x'(\cos^2 \alpha + \sin^2 \alpha) = x \cos \alpha + y \sin \alpha$$

or $$x' = x \cos \alpha + y \sin \alpha .$$

Likewise, $$y' = -x \sin \alpha + y \cos \alpha .$$

4. Since $\alpha = 45^\circ$, the equations of rotation are

$$\begin{cases} x = \frac{1}{\sqrt{2}}(x' - y') \\ y = \frac{1}{\sqrt{2}}(x' + y') \end{cases} .$$

The new equation is $\dfrac{(x' - y')^2}{2} = \dfrac{x' + y'}{\sqrt{2}}$ which simplifies to

$x'^2 - 2x'y' + y'^2 - \sqrt{2}\, x' - \sqrt{2}\, y' = 0$.

5. The solution is similar to that of Exercise (4) . The answers are:

(a) $x'^2 + 5y'^2 = 6$.

(b) (Here $\sin\theta = \frac{\sqrt{5}}{5}$, $\cos\theta = \frac{2\sqrt{5}}{5}$) .

$25x'^2 + 13y'^2 = 25$.

(c) $y'^2 - x'^2 = 8$.

(d) $x'^2 = -4y'$

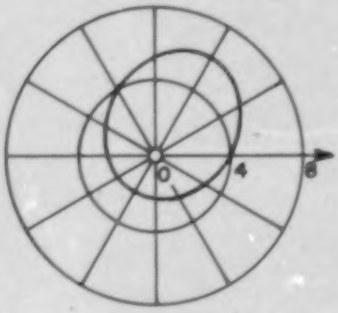
6. After rotation, we have

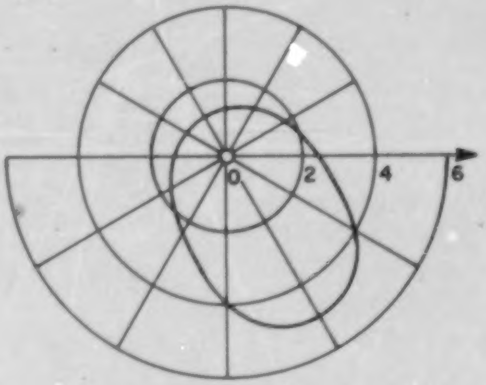
$$(x' \cos\alpha - y' \sin\alpha)^2 + (x' \sin\alpha + y' \cos\alpha)^2 = r^2 ,$$

or
$$x'^2 \cos^2\alpha - 2x'y' \sin\alpha\cos\alpha + y'^2 \sin^2\alpha +$$
$$x'^2 \sin^2\alpha + 2x'y' \sin\alpha\cos\alpha + y'^2 \cos^2\alpha = r^2 .$$

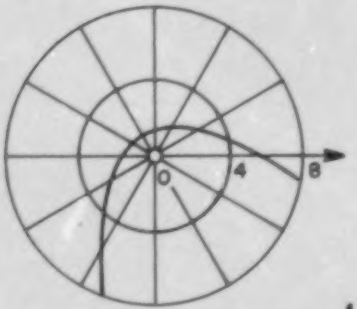
Thus, $x'^2 + y'^2 = r^2$.

7. (a) $r = r\,\dfrac{6}{2 - \cos(\theta - 60^\circ)}$ (b) $r = \dfrac{10}{5 + 3\cos(\theta - 120^\circ)}$





(c) $r = \dfrac{3}{1 + \sin(\theta - 30^\circ)}$



Challenge Problems

1. The proof is as follows: after rotation of axes, the new equation is

$$A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = 0 \quad \text{where}$$

$$A' = A \cos^2 \alpha + B \sin \alpha \cos \alpha + C \sin^2 \alpha$$

$$B' = -2A \sin \alpha \cos \alpha + B \cos^2 \alpha - B \sin^2 \alpha + 2C \sin \alpha \cos \alpha$$

$$C' = A \sin^2 \alpha - B \sin \alpha \cos \alpha + C \cos^2 \alpha$$

$$D' = D \cos \alpha + E \sin \alpha$$

$$E' = -D \sin \alpha + E \cos \alpha$$

$$F' = F$$

When you perform the indicated operations and simplify, you find that

$$B'^2 - 4A'C' = B^2 - 4AC .$$

2. $$x = x' \cos \alpha - y' \sin \alpha = (x'' \cos \theta - y'' \sin \theta) \cos \alpha - (x'' \sin \theta + y'' \cos \theta) \sin \alpha$$

$$x''(\cos \theta \cos \alpha - \sin \theta \sin \alpha) - y''(\sin \theta \cos \alpha + \cos \theta \sin \alpha)$$

$$\therefore \quad x = x'' \cos(\theta + \alpha) - y'' \sin (\theta + \alpha) .$$

Likewise, $y = x'' \sin(\theta + \alpha) + y'' \cos(\theta + \alpha)$

10-4. Invariant Properties.

We have already touched upon the significance of the study of the geometric properties invariant under certain transformations. When the axes are rotated or translated, and the figure remains fixed, the question of invariant properties has meaning only with respect to observers using different points or lines of reference. When we study point transformations in the next two sections, the question of invariant properties has many more and varied aspects. When the points of a figure are moved, we are frequently not certain about the appearance of the image; it may or may not be congruent to the original figure.

You may wish to omit the discussion following Theorem 10-3. It is included to show a second approach to the problem and to lead to an interesting challenge exercise.

The exercises for this section were deliberately selected to point out properties other than distance and angle which remain invariant under the set of translations and rotations. We encourage you to discuss these other properties carefully on the basis of the exercises. The students should be encouraged to find more invariants than are indicated.

Exercises 10-4

1. (a) $3x + 2y - 8 = 0$

   (b) The equations of translation are

$$\begin{cases} x' = x + 4 \\ y' = y + 6 \end{cases} \quad \text{or} \quad \begin{cases} x = x' - 4 \\ y = y' - 6 \end{cases} .$$

   Thus $A = (6,7)$ and $B = (4,10)$ with respect to the new origin. Also, with respect to the new origin, the line has the equation $3(x' - 4) + 2(y' - 6) - 8 = 0$, which simplifies to $3x' + 2y' - 32 = 0$.

   (c) $d(A,B) = \sqrt{13}$ with respect to either set of axes.

2. The equations of rotation are

$$\begin{cases} x' = x \cos \alpha + y \sin \alpha = y \\ y' = -x \cos \alpha + y \cos \alpha = -x \end{cases}$$

   Thus $A = (1,-2)$ and $B = (4,0)$ with respect to the new axes. $d(A,B) = \sqrt{13}$ with respect to either set of axes.

3. The equations of translation are

$$\begin{cases} x' = x + 1 \\ y' = y + 1 \end{cases} \quad \text{or} \quad \begin{cases} x = x' - 1 \\ y = y' - 1 \end{cases} .$$

   (a) $A' = (1,-3)$, $B' = (3, -\frac{1}{3})$ and $C' = (4,1)$.

   L' has the equation $4(x' - 1) - 3(y' - 1) - 12 = 0$, which reduces to $4x' - 3y' - 13 = 0$.

   (b) B is between A and C since $d(A,B) + d(B,C) = d(A,C)$. B' is between A' and C' since $d(A',B') + d(B',C') = d(A',C')$.

   (c) Since $d(A',B') + d(B',C') = d(A',C')$, the points are collinear. Another way to prove collinearity is to show that the slopes of $\overline{A'B'}$, $\overline{B'C'}$, and $\overline{A'C'}$ are equal since B' is common to $\overline{A'B'}$ and $\overline{B'C'}$.

4. (a) The lines are concurrent since the point $(2,1)$ lies on all three lines.

   (b) The equations of translation are:

$$\begin{cases} x' = x - 3 \\ y' = y + 2 \end{cases} \quad \text{or} \quad \begin{cases} x = x' + 3 \\ y = y' - 2 \end{cases} .$$

The equations of the three lines with respect to the new axes are:

$$L_1' : 4x' - 3y' + 13 = 0$$

$$L_2' : x' - 2y' + 7 = 0$$

$$L_3' : 5x' - 3y' + 14 = 0 \ .$$

(c) The lines are concurrent since the point $(-1,3)$ lies on all three lines.

(d) Point $(2,1)$ maps into $(-1,3)$ under this translation.

(e) When the axes are rotated through an angle of 45°, the equations of rotation are

$$\begin{cases} x' = \frac{1}{\sqrt{2}}(x + y) \\ y' = \frac{1}{\sqrt{2}}(-x + y) \end{cases} \quad \text{or} \quad \begin{cases} x = \frac{1}{\sqrt{2}}(x' - y') \\ y = \frac{1}{\sqrt{2}}(x' + y') \ . \end{cases}$$

(1) The equations of the three lines with respect to the new axes are:

$$L_1' : x' - 7y' - 5\sqrt{2} = 0$$

$$L_2' : x' + 3y' = 0$$

$$L_3 : 2x' - 8y' - 7\sqrt{2} = 0$$

(2) The lines are concurrent since the point $\left(\frac{3}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ lies on all three lines.

(3) Point $(2,1)$ maps into $\left(\frac{3}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ under this rotation.

5. (a) $m_1 = -\frac{3}{2}$, $m_2 = 5$

$$\cos\theta = \frac{1 + m_1m_2}{\sqrt{1 + m_1^2}\,\sqrt{1 + m_2^2}} = -\frac{\sqrt{2}}{2}$$

$\therefore \ \theta = \frac{3\pi}{4}$ and its supplement is $\frac{\pi}{4}$.

(b) The equations of translation are

$$\begin{cases} x = x' + 2 \\ y = y' + 2 \,. \end{cases}$$

With respect to the new axes, the equations of the lines are:

$$L_1' : 3x' + 2y' + 2 = 0$$

$$L_2' : 5x' - y' - 1 = 0$$

Since $m_1' = -\frac{3}{2}$, $m_2' = 5$, $\theta' = \frac{\pi}{4}$(and $\frac{3\pi}{4}$) .

Challenge Problem

The proof that the measure of angle is invariant under rotation may be presented as follows:

(1) Consider the angle between the lines

$$L_1 : a_1x + a_2y + a_3 = 0$$

$$L_2 : b_1x + b_2y + b_3 = 0 \,.$$

It will be convenient to use the formula for the angle between two lines in the form: $\tan \alpha = \dfrac{m_2 - m_1}{1 + m_1m_2}$ which is equivalent to the cosine form developed in this text. Thus:

$$\tan \alpha = \frac{\dfrac{-b_1}{b_2} + \dfrac{a_1}{a_2}}{1 + \dfrac{a_1b_1}{a_2b_2}} = \frac{a_1b_2 - a_2b_1}{a_1b_1 + a_2b_2}$$

(2) The equations of rotation are

$$x = x' \cos \theta - y' \sin \theta$$

$$y = x' \sin \theta + y' \cos \theta \,.$$

Substituting in L_1 and L_2, we have

$$L_1' : (a_1 \cos \theta + a_2 \sin \theta)x' + (a_2 \cos \theta - a_1 \sin \theta)y' + a_3 = 0.$$

$$L_2' : (b_1 \cos \theta + b_2 \sin \theta)x' + (b_2 \cos \theta - b_1 \sin \theta)y' + b_3 = 0.$$

The angle between L_1 and L_2 is given by

$$\tan \alpha' = \frac{(a_1 \cos\theta + a_2 \sin\theta)(b_2 \cos\theta - b_1 \sin\theta) - (a_2 \cos\theta - a_1 \sin\theta)(b_1 \cos\theta + b_2 \sin\theta)}{(a_1 \cos\theta + a_2 \sin\theta)(b_1 \cos\theta + b \sin\theta) + (a_2 \cos\theta - a_1 \sin\theta)(b_2 \cos\theta - b_1 \sin\theta)}$$

(3) This complicated expression reduces to

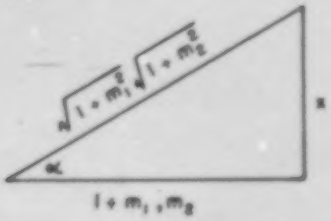
$$\tan \alpha' = \frac{a_1 b_2 - a_2 b_1}{a_1 b_1 + a_2 b_2} = \tan \alpha$$

after several applications of the identity $\sin^2 \alpha + \cos^2 \alpha = 1$.

Thus $\alpha = \alpha'$ for the principal values of $\tan \alpha$ and $\tan \alpha'$.

NOTE: Before offering this problem, you may wish to show the equivalence of the two formulas: $\cos \alpha = \dfrac{1 + m_1 m_2}{\sqrt{1 + m_1^2}\,\sqrt{1 + m_2^2}}$ and $\tan \alpha = \dfrac{m_2 - m_1}{1 + m_2 m_1}$.

The equivalence follows from the Pythagorean Theorem and the definitions of the trigonometric functions.



$$x^2 + (1 + m_1 m_2)^2 = (1 + m_1^2)(1 + m_2^2) .$$

Thus $x = \pm (m_2 - m_1)$ and $\tan \alpha = \dfrac{\pm (m_2 - m_1)}{1 + m_2 m_1}$. This formula gives us the angle and its supplement. We chose $+(m_2 - m_1)$ arbitrarily.

10-5. Point Transformations.

Most of the transformations studied by mathematicians are considered as point transformations or mappings. The reason for dicussing the transformation of axes first is that this type is most useful in reducing a complicated equation to a simpler form for sketching. Considerable care has been taken to distinguish between the two sets of transformations and to indicate that translations and rotations can be effected by either type.

The material included here on reflections relates so closely to the discussion of symmetry in Section 6-2 that a review of that topic may be appropriate before proceeding with this subject.

Euclidean geometry in characterized by the fact that the measures of both distance and angle are preserved under translation and rotation. In elementary geometry this statement is expressed in the "Postulate for Rigid Motion" which states that an object may be moved in space without changing its size and shape. We now see that all rigid motion can be performed by a series of no more than three reflections.

The SMSG Geometry, Appendix 8, has an excellent discussion on rigid motion.

The students may wonder why the third reflection is necessary in Example 3. Two reflections are sufficient if the sequence of points on the line is not considered. You might label several points on $\overline{A'B'}$, see where they fall on $\overline{CD}$, and then observe what happens after the second and third reflections.

An interesting result is obtained by subtracting the corresponding members of the equations of circles C and C' dercribed on this page. The result is $8x + 12y = 0$ or $y = \frac{2}{3}x$. This is the equation of the common chord (or radical axis) shown.

A reflection is an example of what is called an "involutory transformation". A transformation is called an involutory transformation if it has the property that, if repeated once, it produces the identity transformation. This can be written analytically as follows: Let $(x,y) \longrightarrow (x'y') \longrightarrow (x'',y'')$. If $x'' = x$ and $y'' = y$, then the transformation is involutory.

Exercises 10-5

1. (a) $A = (1,2)$ maps into $A' = (1,-2)$ and $B = (3,-4)$ maps into $B' = (3,4)$ after reflection with respect to the x-axis.

$$d(A,B) = 2\sqrt{10} = d(A',B') .$$

(b) $A = (1,2)$ maps into $A' = (-1,2)$ and $B = (3,-4)$ maps into $B'(-3,-4)$ after reflection with respect to the y-axis.

$$d(A,B) = 2\sqrt{10} = d(A',B') .$$

(c) $A = (1,2)$ maps into $A' = (-1,-2)$ and $B = (3,-4)$ maps into $B' = (-3,4)$ after reflection with respect to the origin.

$$d(A,B) = 2\sqrt{10} = d(A',B') .$$

(d) $A = (1,2)$ maps into $A' = (11,2)$ and $B = (3,-4)$ maps into $B' = (9,-4)$ after reflection with respect to the line $x = 6$.

$$d(A,B) = 2\sqrt{10} = d(A',B') .$$

2. We choose the points $A = (2,0)$, $B = (4,0)$, $C = (7,0)$. Under this transformation, the images of these points are $A' = (4,0)$, $B' = (6,0)$, and $C' = (9,0)$. Two invariant properties are the measure of distance and the order of the points on the line. The three points on the line also remain collinear (a third invariant property).

3. Under the transformation $x' = 2x$, the images of the three points are $A' = (4,0)$, $B' = (8,0)$, and $C' = (14,0)$. Three invariant properties are: the origin remains fixed, i.e., $(0,0) \longrightarrow (0,0)$, the order of the three points on the line, and collinearity. (Note that distance is not an invariant property under this transformation.)

4. The angle between L_1 and L_2 has measure 45° . When both lines are rotated through an angle of measure $\frac{\pi}{4}$, the equation of L_1 becomes $y' = x'$ and the equation of L_2 becomes $x' = 0$. The angle between these lines also has measure 45° .

(NOTE: This problem can, of course, be solved by using the equations of rotation. Since the lines are rotated through an angle of measure $\frac{\pi}{4}$, the axes must be rotated through angle of measure $-\frac{\pi}{4}$ to achieve the same result.)

5. The images are:

(a) $y^2 = -x$

(b) $x^2 = -y$

(c) $xy = 6$

(d) $x^2 - y^2 = 1$

(e) $x^2 + y^2 + 2x + 4y + 4 = 0$

(f) $y = -x^3$

(g) $y = -\sin x$

(h) $y = -\tan x$

(i) $y = 2^{-x}$

It is recommended that the graph for the original curve and its image be drawn on the same set of axes.

6. In this problem the points are rotated about the origin through an angle θ such that $\tan\theta = \frac{3}{4}$. In order to achieve the same result, we rotate the axes through an angle θ such that $\tan\theta = \frac{-3}{4}$. Thus $\sin\theta = -\frac{3}{5}$, $\cos\theta = \frac{4}{5}$, and the equations of rotation are

$$\begin{cases} x' = \frac{1}{5}(4x - 3y) \\ y' = \frac{1}{5}(3x + 4y) \end{cases} .$$

Under this rotation $A = (-2,1)$ maps into $A' = (-\frac{11}{5}, -\frac{2}{5})$, $B = (5,-2)$ maps into $B' = (\frac{26}{5}, \frac{7}{5})$, and $C = (3,3)$ maps into $C' = (\frac{3}{5}, \frac{21}{5})$.

Invariant properties are:

(a) Measure of Distance. For example, $d(A,B) = \sqrt{58} = d(A',B')$.

(b) Measure of Angle. For example,

$m_{\overline{AB}} = -\frac{3}{7}$, $m_{\overline{AC}} = \frac{2}{5}$, $\cos A = \frac{1}{\sqrt{2}}$, and $m\angle A = \frac{\pi}{4}$.

$m_{\overline{A'B'}} = \frac{9}{37}$, $m_{\overline{A'C'}} = \frac{23}{14}$, $\cos A' = \frac{1}{\sqrt{2}}$, and $m\angle A' = \frac{\pi}{4}$.

(c) Area of $\triangle ABC$ = Area of $\triangle A'B'C'$.

Apply the formula $s = \sqrt{s(s-a)(s-b)(s-c)}$ where s is the semi-perimeter.

7. We do not present the constructions here since the procedure is shown in the text. In part (b), each corresponding pair of lines must be mapped separately.

8. Since the points on the curves are rotated through an angle of measure $\frac{\pi}{6}$, the axes must be rotated through an angle of measure $-\frac{\pi}{6}$. The equations of rotation are, therefore:

$$\begin{cases} x = \frac{1}{2}(\sqrt{3}\ x' + y') \\ y = \frac{1}{2}(-x' + \sqrt{3}\ y') . \end{cases}$$

(a) The image of the line $3x + 2y - 8 = 0$ is
$\frac{3}{2}(\sqrt{3}\ x' + y') + \frac{2}{2}(-x' + \sqrt{3}\ y') - 8 = 0$, which simplifies to
$(3\sqrt{3} - 2)x' + (2\sqrt{3} + 3)y' - 16 = 0$.

(b) The image of the circle $x^2 + y^2 = 25$ is $x'^2 + y'^2 = 25$.

(c) The image of the parabola $y^2 = 4x$ is
$x'^2 - 2\sqrt{3}x'y' + 3y'^2 - 8\sqrt{3}x' - 8y' = 0$.

(NOTE: You may wish to excuse your students from sketching this parabola.)

9. Another way to write this transformation is

$$\begin{cases} x' = -y + 3 \\ y' = x + 1 \end{cases} \quad \text{or} \quad \begin{cases} x = y' - 1 \\ y = -x' - 3 . \end{cases}$$

The images of the curves in Exercise 8 are:

(a) The image of the line $3x + 2y - 8 = 0$ is
$3(y' - 1) + 2(-x' - 3) - 8 = 0$, which simplifies to
$2x' - 3y' + 17 = 0$. Note that these lines are perpendicular.

(b) The image of the circle $x^2 + y^2 = 25$ is the circle
$x'^2 + y'^2 + 6x' - 2y' - 15 = 0$ which has its center at $(-3,1)$ and a radius of 5.

(c) The image of the parabola $y^2 = 4x$ is $x'^2 + 6x' - 4y' + 13 = 0$.

10. Another way to write this transformation is:

$$\begin{cases} x' = x + y \\ y' = 2x - y \end{cases} \quad \text{or} \quad \begin{cases} x = \frac{1}{3}(x' + y') \\ y = \frac{1}{3}(2x' - y') . \end{cases}$$

The line $L_1 : 3x - 2y + 5 = 0$ maps into the line

$L_1' : x' - 5y' - 15 = 0$.

The line $L_2 : 3x - 2y - 3 = 0$ maps into the line

$L_2' : x' - 5y' + 9 = 0$.

$L_1' \| L_2'$ since they have the same slope.

10-6. Inversions.

The justification for Section 6 was presented in Section 10-1 and in Section 10-5 of this commentary. This transformation has been studied by many outstanding mathematicians and plays a role in the Poincaré model of non-Euclidean geometry. You will find an excellent discussion of inversion geometry in Introduction to Higher Geometry by William C. Graustein.

An inversion is an involutory transformation as defined in Section 10-5 of this commentary. If an inversion T carries $P \rightarrow P'$, a second application of T will carry $P' \rightarrow P$.

You may like to point out to the class that as point P approaches the origin, the image P' will recede farther and farther out in the plane. For this reason it is often stated that the center of the circle of inversion corresponds to the "point at infinity" under the inversion. This is a useful concept since we can now say that an inversion sets up a one-to-one correspondence between the points of the plane and their images.

One of the most important properties of an inversion is that it transforms straight lines and circles into straight lines and circles. Specifically, we show that, after an inversion:

1. A line through the origin inverts into the same line through the origin, although the points on the line are interchanged.
2. A line not through the origin inverts into a circle through the origin.

3. A circle through the origin inverts into a straight line not through the origin.
4. A circle not through the origin inverts into a circle not through the origin.

You may want to precede Example 1 by a similar problem wherein the constants a , b , c are specified. You could then draw the unit circle of inversion, the straight line, and its inverse on the same set of coordinates. You can thus verify that the inverse really does pass through the origin.

This same comment holds for Examples 2 and 3 . In Example 2, let $x = \pm 1$, $x = \pm 2$, $y = \pm 1$, $y = \pm 2$ and observe what happens. It may also be profitable and interesting to explore with your class the inverses of a family of circles concentric to the unit circle; for example $x^2 + y^2 = 4$, $x^2 + y^2 = 9$, ... This could be followed by a set of concentric circles with their centers at $(2,4)$ or some other point.

If you have studied projective geometry or non-Euclidean geometry you undoubtedly recall the cross-ratio which appears in Exercise 9. The cross-ratio is invariant under a projective transformation and certain other transformations as well as under the inversion transformation. This property plays a very important role in the proof of the consistency of non-Euclidean geometry. If interested, you may like to read Chapter 4 of Foundations and Fundamental Concepts of Mathematics by Newsom and Eves.

Exercises 10-6

1. The inverse of the line $3x + 2y - 6 = 0$ is

$$\frac{3x'}{x'^2 + y'^2} + \frac{2y'}{x'^2 + y'^2} - 6 = 0 ,$$

which simplifies to $x'^2 + y'^2 - \frac{x'}{2} - \frac{y'}{3} = 0$. This represents a circle with center at $(\frac{1}{4}, \frac{1}{6})$ and radius $\frac{\sqrt{13}}{12}$. The circle passes through the origin.

2. The inverse of the line $y = 5x$ is the line $y' = 5x'$. The line inverts into itself.

3. The inverse of the line $y = 3$ is the curve $\frac{y'}{x'^2 + y'^2} = 3$, which simplifies to $x'^2 + y'^2 - \frac{1}{3}y' = 0$. It is a circle which has its center at $(0,\frac{1}{6})$ and radius $\frac{1}{6}$. The circle passes through the origin.

4. The inverse of the parabola $y^2 = 4x$ is the curve

$$\frac{y'^2}{(x'^2 + y'^2)^2} = \frac{4x'}{(x'^2 + y'^2)}, \text{ which simplifies to } y'^2 = \frac{4x'^3}{1 - 4x'}.$$

NOTE: The graph of this curve may be left as a challenge exercise. It is a cissoid with the following properties:

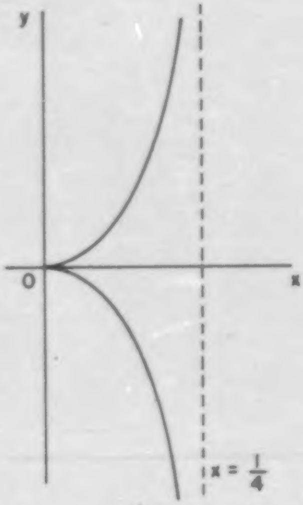
(1) Symmetry with respect to the x-axis.

(2) Intercept at $(0,0)$.

(3) Asymptote: $x' = \frac{1}{4}$.

(4) Extent: $0 < x' < \frac{1}{4}$.

The curve has this appearance



5. The inverse of the circle $(x - 4)^2 + (y - 4)^2 = 16$ is found as follows: The equation simplifies to $x^2 + y^2 - 8x - 8y + 16 = 0$. Apply the transformation and obtain:

$$\left(\frac{x'^2}{(x'^2 + y'^2)^2} + \frac{y'^2}{(x'^2 + y'^2)^2}\right) - \frac{8x'}{x'^2 + y'^2} - \frac{8y'}{x'^2 + y'^2} + 16 = 0$$

or $$\frac{1}{x'^2 + y'^2} - \frac{8x'}{x'^2 + y'^2} - \frac{8y'}{x'^2 + y'^2} + 16 = 0$$

or $1 - 8x' - 8y' + 16(x'^2 + y'^2) = 0$.

or $x'^2 + y'^2 - \frac{x'}{2} - \frac{y'}{2} + \frac{1}{16} = 0$ which represents a circle with center at $(\frac{1}{4}, \frac{1}{4})$ and radius $\frac{1}{4}$.

6. This problem is essentially solved in Example 2 of the text.

7. The inverse of the line $L : 3x + 2y - 6 = 0$ is the circle

$L' : x'^2 + y'^2 - \frac{x'}{2} - \frac{y'}{3} = 0$.

We now apply the inverse transformation

$$\begin{cases} x' = \dfrac{x}{x^2 + y^2} \\ y' = \dfrac{y}{x^2 + y^2} \end{cases} \quad \text{to } L' \text{ and obtain:}$$

$$\left(\frac{x^2}{(x^2 + y^2)^2} + \frac{y^2}{(x^2 + y^2)^2}\right) - \frac{x}{2(x^2 + y^2)} - \frac{y}{3(x^2 + y^2)} = 0 ,$$

or $$\frac{1}{x^2 + y^2} - \frac{x}{2(x^2 + y^2)} - \frac{y}{3(x^2 + y^2)} = 0 ,$$

or $3x + 2y - 6 = 0$, which we recognize as the original line L .

A reasonable conjecture is that a second application of the same inverse transformation yields the original curve. This verifies the fact that an inversion is an involutory transformation.

8. If we had not used a unit circle, we would have had $d(O,P) \cdot d(O,P') = r^2$

or $d(O,P) = \frac{r^2}{d(O,P')}$.

Since $\frac{d(O,P)}{d(O,P')} = \frac{x}{x'} = \frac{r^2}{(d(O,P'))^2} = r^2(d(O,P))^2$,

we have $x = \frac{r^2 x'}{x'^2 + y'^2}$ and $x' = \frac{r^2 x}{x^2 + y^2}$.

Likewise, $y = \frac{r^2 y'}{x'^2 + y'^2}$ and $y' = \frac{r^2 y}{x^2 + y^2}$.

9. Since $r = 2$, the inverse transformation is :

$$\begin{cases} x' = \dfrac{4x}{x^2 + y^2} \\ y' = \dfrac{4y}{x^2 + y^2} \end{cases}$$

The inverse points are as follows:

$$A = (0,-3) \longrightarrow A' = (0, -\tfrac{4}{3})$$

$$B = (1,-1) \longrightarrow B' = (2,-2)$$

$$C = (2,1) \longrightarrow C' = (\tfrac{8}{5}, \tfrac{4}{5})$$

$$D = (3,3) \longrightarrow D' = (\tfrac{2}{3}, \tfrac{2}{3})$$

$d(A,C) = 2\sqrt{5}$, $d(A,D) = 3\sqrt{5}$, $d(B,C) = \sqrt{5}$, $d(B,D) = 2\sqrt{5}$.

$$\frac{d(A,C)}{d(A,D)} \div \frac{d(B,C)}{d(B,D)} = \frac{2\sqrt{5}}{3\sqrt{5}} \div \frac{\sqrt{5}}{2\sqrt{5}} = \frac{4}{3} .$$

$d(A',C') = \frac{8}{3}$, $d(A',D') = \frac{2}{3}\sqrt{10}$, $d(B',C') = 2\sqrt{2}$, $d(B',D') = \frac{4}{3}\sqrt{5}$.

$$\frac{d(A',C')}{d(A',D')} \div \frac{d(B',C')}{d(B',D')} = \frac{\frac{8}{3}}{\frac{2}{3}\sqrt{10}} \div \frac{2\sqrt{2}}{\frac{4}{3}\sqrt{5}} = \frac{4}{3} .$$

The above verifies that an inversion is a cross-ratio preserving transformation.

10. The instructions for the construction are given in the text. As in the proof in the text for the first construction:

$$\angle ORP = \angle POR = \angle OPR' .$$

Thus $\triangle RP'O \sim \triangle ORP$ and

$$\frac{d(0,P')}{d(0,R)} = \frac{d(0,R)}{d(0,P)} \text{ or } d(0,P) \cdot d(0,P \text{ [illegible] } = r^2 .$$

Review Exercises

Before proceeding with the solutions of this set of problems, we feel it important to point out that there are two ways to interpret the mapping symbol, such as $(x,y) \longrightarrow (2x,3y)$, which appears in the first exercise.

In this text, we have adopted the convention that $(x,y) \longrightarrow (2x,3y)$ is merely another way of writing the transformation

$$\begin{cases} x' = 2x \\ y' = 3y , \end{cases} \qquad \text{or} \qquad \begin{cases} x = \frac{x'}{2} \\ y = \frac{y'}{3} . \end{cases}$$

A second interpretation, which is often used (but not in this text), is the following: wherever an x appears in the equation, replace it by $2x$; wherever a y appears, replace it by $3y$.

The first interpretation leads to the result $x'^2 = \frac{8}{3}y'$; the second interpretation leads to the result $x^2 = \frac{3}{2}y$ when applied to Exercise 1(a).

Solutions

1. (a) $x'^2 = \frac{8}{3}y'$ (See above)

   (b) The transformation can be written as

$$\begin{cases} x' = x + 2 \\ y' = 3y \end{cases} \qquad \text{or} \qquad \begin{cases} x = x' - 2 \\ y = \frac{y'}{3} . \end{cases}$$

Applying this transformation to the parabola $x^2 = 2y$, we have $(x' - 2)^2 = 2\frac{y'}{3}$, which simplifies to $3x'^2 - 12x' - 2y' + 12 = 0$. An invariant property of this transformation is that a parabola maps into a parabola; i.e., the type of curve is invariant.

   (c) The transformation can be written as

$$\begin{cases} x' = x - 1 \\ y' = y + 2 \end{cases} \qquad \text{or} \qquad \begin{cases} x = x' + 1 \\ y = y' - 2 . \end{cases}$$

The parabola $x^2 = 2y$ transforms into the curve:

$(x' + 1)^2 = 2(y' - 2)$ which we recognize as the equation of the same parabola with respect to a new origin at $(-1,2)$.

2. The mapping $(x,y) \longrightarrow (kx,ky)$ can be written as the transformation

$$\begin{cases} x' = kx \\ y' = ky \end{cases} \quad \text{or} \quad \begin{cases} x = \frac{x'}{k} \\ y = \frac{y'}{k} \end{cases}$$

In this case, we let $k = 2$. Applying this transformation, in turn, to each of the curves, we have:

(a) The line $2x + 3y - 6 = 0$ transforms to $2x' + 3y' - 12 = 0$, a line parallel to the original line.

(b) The circle $x^2 + y^2 = 25$ transforms to $x'^2 + y'^2 = 100$, a circle with the same center but with a radius twice as large.

(c) The parabola $y^2 = -4x$ transforms to the parabola $y'^2 = -8x$, which is "parallel" to the original curve.

The title is well justified since a figure "similar" to the original appears after the transformation.

3. Under this transformation, the image of L_1 is the line $L_1' : x - 5y - 8 = 0$, and the image of L_2 is the line $L_2' : 5x + y - 12 = 0$. $L_1 \perp L_2$ since $m_1' \cdot m_2' = -1$.

4. Applying the transformation T to each of the curves, we obtain the following images:

(a) $13x'^2 - 4x'y' + 20y'^2 - 20x' - 24y' + 13 = 0$. (An ellipse)

(b) $65x'^2 + 172x'y' - 28y'^2 + 232x' + 112y' - 176 = 0$. (A hyperbola)

(c) $x + 22y - 28 = 0$.

(d) $x + 22y - 17 = 0$.

Lines (c) and (d) are parallel and their images are parallel. A reasonable conjecture is that an affine transformation preserves the property of parallelism. (It does.)

5. The proof follows that given in the text for the mapping $(x,y) \longrightarrow (x,-y)$ in Section 10-5.

Supplement to Chapter 2

Exercises S2-1a

1. $p' = 8$ $q' = 0$ $r' = -4$
scale preserving, order reversing

2. $p' = -22$ $q' = 10$ $r' = 26$
scale decreasing, order preserving

3. $p' = -1$ $q' = 1$ $r' = 2$
scale increasing, order preserving

4. $p' = 15$ $q' = -9$ $r' = -21$
scale decreasing, order reversing

5. $p' = \frac{17}{3}$ $q' = \frac{1}{3}$ $r' = -\frac{7}{3}$
scale increasing, order reversing

6. $p' = 2$ $q' = 10$ $r' = 14$
scale preserving, order preserving

7. Let P be the origin point, Q the unit point in the original system; i.e., $p = 0$ $q = 1$.

(5) $p' = 3$ $q' = 2$
(6) $p' = -2$ $q' = 2$
(7) $p' = \frac{1}{4}$ $q' = \frac{1}{2}$
(8) $p' = 0$ $q' = -3$
(9) $p' = \frac{7}{3}$ $q' = \frac{2}{3}$
(10) $p' = 7$ $q' = 8$

8. Let P be the origin of the new system, Q the new unit point; i.e.,

$p' = 0 \qquad q' = 1$

(5) $p = 3 \qquad q = 2$
(6) $p = \frac{1}{2} \qquad q = \frac{3}{4}$
(7) $p = -1 \qquad q = 3$
(8) $p = 0 \qquad q = -\frac{1}{3}$
(9) $p = \frac{7}{2} \qquad q = 2$
(10) $p = -7 \qquad q = -6$

9. Suppose $a = 0$ in $x' = ax + b$.

Then for any point P with coordinate p, we would get $p' = 0 \cdot p + b$. So every point in the new system would have coordinate b, thus preserving neither measure nor order nor betweenness.

10. $x' = ax^3 + b$

Let p and p' be the intrinsic and the new coordinates of P, and similarly for q and q', etc. Let $d'(P,Q) = |p' - q'|$. Then

$$d'(P,Q) = |ap^3 + b - aq^3 - b| = |a|\,|p^3 - q^3|$$
$$= |a|\,|p - q|\,|p^2 + pq + q^2|.$$

Similarly

$$d'(R,S) = |a|\,|r - s|\,|r^2 + rs + s^2|.$$

Suppose $\overline{PQ} \cong \overline{RS}$. Then $|p - q| = |r - s|$. However, $d'(P,Q) = d'(R,S)$ only if $|p^2 + pq + q^2| = |r^2 + rs + s^2|$, which in general is false. For example, if $p = 1$, $q = 2$, $r = 3$ and $s = 4$, then $|p^2 + pq + q^2| = 7$ while $|r^2 + rs + s^2| = 37$. It is also true that we can have $d'(P,Q) = d'(R,S)$ although $\overline{PQ}$ and $\overline{RS}$ are not congruent. The example $p = 0$, $q = \sqrt[3]{7}$, $r = 1$ and $s = 2$ shows this. $p < q < r$ always implies $p^3 < q^3 < r^3$, so betweenness is preserved.

11. $x' = e^x$

$$d'(P,Q) = |e^p - e^q|$$
$$d'(R,S) = |e^r - e^s|$$

So $\overline{PQ} \cong \overline{RS}$ does not always imply $d'(P,Q) = d'(R,S)$

$p < q < r$ does always imply $e^p < e^q < e^r$, so betweenness is preserved.

12. $x' = \frac{1}{x}$ if $x \neq 0$

$x' = x$ if $x = 0$

If none of p, q, r, s is zero

$$d'(P,Q) = \frac{1}{|pq|}|p - q|$$

$$d'(R,S) = \frac{1}{|rs|}|r - s|$$

So $PQ \cong RS$ does not always imply $d'(P,Q) = d'(R,S)$. However, if $P = R = 0$ and $\overline{PQ} \cong \overline{RS}$, then $|q| = |s|$ and $d'(P,Q) = d'(R,S)$. Let $p < q < r$. Then betweenness is preserved only if $q = 0$ or $r < 0$ or $p > 0$.

13. $x' = \log_{10} x$

This cannot handle points on negative side of the origin since $\log_{10}$ is not defined for negative numbers or 0. Where it is defined

$$d'(P,Q) = |\log_{10} \frac{p}{q}|$$

$$d'(R,S) = |\log_{10} \frac{r}{s}|$$

So $\overline{PQ} \cong \overline{RS}$ does not always imply $d'(P,Q) = d'(R,S)$. Betweenness is preserved where $\log_{10}$ is defined.

The notion of a group will mean very little to the students unless they consider many examples. They should study carefully all those mentioned in the text and try to think of others. If they know something about complex numbers, they can be asked to prove that the three cube roots of 1 form a group under multiplication, as do the four fourth roots. These examples show that a group may be finite. If the students are asked for other finite groups, some of them may suggest the kind of arithmetic that suits clock faces. Finally, no complicated mathematical definition becomes clear to students until they have thought of examples that don't quite fit. What about the integers under multiplication, the non-negative integers under addition, and the rational numbers under multiplication?

Exercises S2-1b

1. Let f be the function defined by $f(x) = ax + b \quad a \neq 0.$
   Let g be the function defined by $g(x) = cx + d \quad c \neq 0.$

   We wish to prove $f(g)$ is a function defined by $(f(g))(x) = sx + t$ for real numbers $s \neq 0$, and t.

   $$\begin{aligned}(f(g))(x) &= f(g(x)) \\ &= a(cx + d) + b \\ &= (ac)x + (ad + b)\end{aligned}$$

   Since $a \neq 0$ and $c \neq 0$ we know that $(ac) \neq 0$.
   Thus there do exist real numbers $s = ac \neq 0$, $t = ad + b$ such that $(f(g))(x) = sx + t$.

2. Consider f, g, h as three functions in our set:

   $$f(x) = mx + n, \quad g(x) = px + q, \quad h(x) = rx + s \qquad m, p, r \neq 0$$

   We wish to show $(f(g))(h) = f(g(h))$
   We find that $f(g)$ is defined by $(f(g))(x) = (mp)x + (mq + n)$ and that $g(h)$ is defined by $(g(h))(x) = (pr)x + (ps + q)$

   Then for all x $(f(g))(h)(x) = (mp)rx + (mp)s + mq + n$
   for all x $f(g(h))(x) = m(pr)x + m(ps + q) + n$
   Hence for all x $(f(g))(h)(x) = (f(g(h)))(x)$ which is the necessary and sufficient condition that the functions S, or for each x, $(f(g))(h) = (f(g(h)))$.

   Note: this is a special case of the theorem that if h maps set A into set B, g maps B into C, and, f maps C into D then $(f(g))(h) = f(g(h))$. The general proof follows: If $x \in A$, let $g = h(x) \in B$, $f = g(y) \in C$, and $f(z) \in D$. Let $k = f(g)$ mapping B into D, $\ell = g(h)$ mapping A into C. Then $(f(a))(h)(x) = k(h)(x) = k(h(x)) = k(y)$ but $k(y) = f(g)(y) = f(y)$. Also $f(g(h))(x) = (f(\ell))(x) = f(z)$ since $(x) = g(h)(x) = g(y) = z$. Therefore $(f(g))(h) = f(g(h))$.

3. Let f be defined by $f(x) = ax + b \quad a \neq 0$

g be defined by $g(x) = cx + d \quad c \neq 0.$

Then $(f(g))(x) = f(cx + d) = a(cx + d) + b = (ac)x + (ad + b)$

$(g(f))(x) = g(ax + b) = (ca)x + (cb + d)$

$f(g) = g(f)$ only if $ad + b = cb + d$.

To show that the commutative property does not hold, we need simply exhibit one case when it doesn't. Take $a = 1, \ c = 2, \ d = 1, \ b = 1;$ then $ad + b = 1 \cdot 1 + 1 = 2 \qquad cb + d = 2 \cdot 1 + 1 = 3$

$(f(g))(x) = 2x + 2 \qquad (g(f))(x) = 2x + 3 \qquad f(g) \neq g(f)$

4. To show that in any group the identity is unique.

Let e and e' be identity elements.

Then for all a, $a(e) = e(a) = a \quad (1)$

$a(e') = e'(a) = a \quad (2)$

So in particular $e'(e) = e(e') = e'$ from (1) letting $a = e'$

$e(e') = e'(e) = e$ from (2) letting $a = e$

Which gives us $e = e'$.

5. To show that in any groups G the inverse is unique. Let $a \in G$. Suppose b and b' are both inverses; i.e.,

$a(b) = b(a) = e$

$a(b') = b'(a) = e$

Now consider $b(a(b')) = (b(a))(b')$ by associativity; but

$b(a) = e$ and $a(b') = e$ so

$b(e) = e(b')$;

but e is the identity element, so

$b = b'$.

6. To show that the inverse of the identity is the identity, let e be the identity, a its inverse.

Then $a(e) = e$ since a is the inverse of e,

but $a(e) = a$ since e is the identity, therefore $a = e$.

7. (a) $a^2x + ab + b$

(b) $apx + aq + b$

(c) $apx + bp + q$

(d) $p^2x + pq + q$

(e) $a^3x + a^2b + ab + b$

(f) $p^3x + p^2q + pq + q$

(g) $\frac{1}{p}x - \frac{q}{p}$

(h) $\frac{1}{ap}x - \frac{q + bp}{ap}$

(i) $\frac{1}{ap}x - \frac{b + aq}{ap}$

(j) $\frac{1}{ap}x - \frac{b + aq}{ap}$

(k) $\frac{p}{a}x - \frac{bp}{a} + q$

(ℓ) $\frac{a}{p}x - \frac{aq}{p} + b$

* 8. Let f be defined by $f(x) = ax + b \quad a \neq 0$.

If $h(h) = f$ we must have $p \neq 0$ and q such that
$h(h(x)) = p^2x + pq + q = ax + b = f(x)$

Thus p and q must satisfy

$$p^2 = a \qquad pq + q = b$$

Case 1. $a < 0$

There is no real number whose square is negative so there is no function h such that $h(h) = f$.

Case 2. $a > 0$ and $a \neq 1$

Both $p = \sqrt{a}$ and $p = -\sqrt{a}$ satisfy $p^2 = a$. So we have, in general, two solutions to $h(h) = f$.

h_1 defined by $h_1(x) = \sqrt{a}\,x + \frac{b}{1 + \sqrt{a}}$

h_2 defined by $h_2(x) = -\sqrt{a}\,x + \frac{b}{1 - \sqrt{a}}$

h_1 is defined for all values of $a \neq 0$ and b. However, in the special case $a = 1$, h_2 is not defined because $1 - \sqrt{a} = 1 - 1 = 0$. So when $a = 1$ we get the unique solution $h(x) = x + \frac{b}{2}$.

Although Section S2-2 can be omitted without serious loss of continuity, there are a good many ideas in it which are important in other branches of mathematics. If you do not think there is time to cover it in class, perhaps the better students could study it and do some of the exercises.

In earlier courses, students have studied various number systems and learned to consider them as sets closed under certain operations but not under others. The fundamental operations of addition and multiplication are commutative. In the set of linear transformations of a line onto itself we have an algebraic operation whose elements are not numbers but functions. The only operation--composition of functions--is not commutative. Nevertheless, the operation is associative. There is an element which plays the same role for composition as zero does for addition and one for multiplication. For each linear transformation there is a transformation which "undoes" the first, and thus acts like the reciprocal of a nonzero number when the operation is multiplication and like the negative of a number when the operation is addition.

It is the fact that so many different algebraic systems share these properties that led mathematicians to define a group. This concept was defined earlier, and the example treated here is one which is very important in advanced mathematics.

If the exercises on cardinal number are to be assigned, it will probably be necessary to prepare the way with a brief discussion in class. It can be pointed out that when we are asked whether two finite sets have the same number of members, we can count them. Now counting a set can be described as setting up a one-to-one correspondence between the set and part of a standard sequence of noises. If we do this for sets A and B and discover that we used the same part of the standard sequence of noises in both cases, we have set up a one-to-one correspondence between A and B. We could have done this without counting. Since we can't, in any ordinary sense, count the members of an infinite set, it is natural to define what we mean when we say that two such sets have the same number of members in terms of one-to-one correspondences. Although the students will probably be a bit disturbed by the fact that the set of positive integers and the set of odd positive integers have the same number of members, they will soon come to realize that no other definition seems reasonable.

The students should be asked to give detailed proofs, in class, for one or two cases of the theorem that an image is between two other images if and only if its pre-image is between the pre-images of the other two images. This will prepare them for the first exercise in the next set. Since we are dealing with a necessary and sufficient condition, two implications must be proved. The proof can be shortened, however, by noting that the inverse of a transformation of any of the four types is of the same type.

Exercises 3-6 of the following set justify that the linear transformation of a line onto itself forms a group under the operation of composition.

Exercises S2-2a

1. Let Q be between P and R; i.e., either $p < q < r$ or $p > q > r$ where p, q, r are coordinates of P, Q, R on line $\overleftrightarrow{PR}$. If T is a linear transformation, then there are numbers $a \neq 0$ and b such that the coordinate of $T(X) = ax + b$ where x is the coordinate of X.

$$T(P) \sim p' = ap + b \qquad T(Q) \sim q' = aq + b \qquad T(R) \sim r' = ar + b$$

If $p < q < r$ and $a > 0$ then $ap < aq < ar$ and $p' < q' < r'$
If $p < q < r$ and $a < 0$ then $ap > aq > ar$ and $p' > q' > r'$
If $p > q > r$ and $a > 0$ then $ap > aq > ar$ and $p' > q' > r'$
If $p > q > r$ and $a < 0$ then $ap < aq < ar$ and $p' < q' < r'$

Hence in all cases $T(Q)$ is between $T(P)$ and $T(R)$.

2. Let $\overline{PQ}$ and $\overline{RS}$ be congruent segments; i.e., $|p - q| = |r - s|$. Let T be a linear transformation, defined: $T(X) = X'$ has coordinate $x' = ax + b$.

$$T(P) \sim p' = ap + b \quad T(Q) \sim q' = aq + b \quad |p'-q'| = |ap+b-aq-b| = |a|\ |p-q|$$
$$T(R) \sim r' = ar + b \quad T(S) \sim s' = as + b \quad |r'-s'| = |ar+b-as-b| = |a|\ |r-s|$$

But $\overline{PQ} \cong \overline{RS}$ implies $|p - q| = |r - s|$. So $|p' - q'| = |r' - s'|$ which means $\overline{P'Q'} \cong \overline{R'S'}$.

3. Let T_1, T_2 be arbitrary linear transformations of the line into itself defined by coordinate equations: $T_1(X) = X'$ $x' = ax + b$, $T_2(X) = X'$ $x' = cx + d$. We wish to know whether $T_1(T_2)$ is a linear transformation of the line.

$T_2(X)$ is a point Y with coordinate $cx + d$

T_1 is defined at Y; $T_1(y)$ is a point with coordinates $(ac)x + (ad + b)$.

But $ac \neq 0$ since $a \neq 0$ and $c \neq 0$. And $(ad + b)$ is a number. So $T_1(T_2)$ is defined for all points X by coordinate equation $x' = (ac)x + (ad + b)$. Thus it is a linear transformation of the line.

4. To show that composition of linear transformations is associative let T_1, T_2, T_3 be defined by coordinate equations $T_1(x) = ax + b$, $T_2(x) = cx + d$, $T_3(x) = ex + f$. Then $T_2(T_3)$ is the linear transformation taking x to $(ce)x + (cf + d)$ and $T_1(T_2)$ is the linear transformation taking x to $(ac)x + (ad + b)$. Let X_0 be an arbitrary point with coordinate x_0.

$T_3(X_0) = Y$ with coordinate $(ex_0 + f)$,

$(T_1(T_2))(Y) = Z$ with coordinate $(ac)(ex_0 + f) + (ad + b)$.

So $((T_1(T_2))T_3)(X_0) = Z$ with coordinate $(ace)x_0 + (acf + ad + b)$.

Now $(T_2(T_3))(X_0) = V$ with coordinate $v = (ce)x_0 + (cf + d)$,

$T_1(V) = Z'$ with coordinate $a((ce)x_0 + (cf + d)) + b$.

So $(T_1(T_2(T_3)))(X_0) = Z'$ with coordinate $(ace)x_0 + (acf + ad + b)$.

Therefore $Z = Z'$ since both have the same coordinate which means

$$T_1(T_2(T_3)) = (T_1(T_2))(T_3).$$

5. To show that the set of linear transformations of a line has an identity with respect to composition, consider line $\overleftrightarrow{OU}$ and the transposition I such that $I(X) = X$. I is given by the coordinate equation $I(x) = x = 1 \cdot x + 0$ so I is a member of the set of linear transformations. This I is an identity. By the definition of I we know

$$(I(T))(X) = I(T(X)) = T(X)$$

or

$$(T(I))(X) = T(I(X)) = T(X)$$

so

$$I(T) = T(I) = T$$

Suppose I' were any other identity.

Then $I'(I) = I(I') = I$ since I' is an identity,

but $I(I') = I'(I) = I'$ since I is an identity.

Therefore $I' = I$ which means I is the unique identity.

6. To show that each element on the set S of linear transformations of the line has an inverse with respect to composition, let T be an arbitrary element of $S \cdot T(X)$ is the point Y such that $y = asc + b$, $b \neq 0$.

If there were an inverse T^{-1} to T we would have to have

$$T^{-1}(T) = T(T^{-1}) = I.$$

There would have to be numbers $c \neq 0$ and d, such that for all points S, with coordinate x,

$$c(ax + b) + d = a(cx + d) + b = 1x + 0.$$

This requires $$cax = acs = 1x \qquad (1)$$
$$cb + d = ad + b = 0 \qquad (2)$$

Since $a \neq 0$ we can choose $c = \frac{1}{a} \neq 0$ to satisfy (1) and then $d = -b$ along with $c = \frac{1}{a}$, $y - b$ will be the inverse of T, and is a linear transformation.

7. We exhibit one counter example to show that composition is not commutative. Consider

$$T_1 : T_1(X) = Y, \quad y = 2x + 0 \text{ [":" is read "defined by"]}$$
$$T_2 : T_2(X) = Y, \quad y = 1 \cdot x + 1$$
$$T_1(T_2) : (T_1(T_2))(X) = 2(x + 1) + 0 = 2x + 2$$
$$T_2(T_1) : (T_2(T_1))(X) = 1(2x + 0) + 1 = 2x + 1$$

Therefore $T_2(T_1) \neq T_1(T_2)$.

Suppose we require $$T_1 : T_1(X) = Y, \quad y = ax + b \text{ and}$$
$$T_2 : T_2(X) = Y, \quad y = cx + d$$

to be such that $T_1(T_2) = T_2(T_1)$, i.e., $a(cx + d) + b = c(ax + b) + d, \forall x$.

So we must have $acx = cax$ and $ad + b = cb + d$.

The conditions are (1) $a = c = 1$ and b and d any real numbers.
(2) $a = c \neq 1$ and $b = d$ any real number.
(3) a, c any real numbers and $b = d = 0$.

8. Let $F : F(X) = Y$, $y = ax + b$ be a transformation.

Case (1) $a > 0$. $F = T(E)$ where $E : y = ax$ $T : y = x + b$

$\forall X$, $E(X)$ has coordinate ax, $T(E(X))$ has coordinate $ax + b$.

Case (2) $a < 0$. $F = T(E(R))$ where $R : y = -1x$ $E : y = |a|x$ $T : y = x + b$

$\forall X$, $R(X)$ has coordinate $-x$, $E(R(X))$ has coordinate $|a|(-x) = ax$

$T(E(R(X)))$ has coordinate $ax + b$ hence $T(E(R)) = F$.

Exercises 2-2b

1. Let the points be R and S. We may assume $r < s$. The ratio of two non-zero numbers is positive if and only if both numbers have the same sign. $r < s$ means $r - s < 0$. Therefore $\frac{r' - s'}{r - s} > 0$ if and only if $r' - s' < 0$. But we have $r' - s' < 0$ if and only if $r' < s'$ which is the condition that the coordinate change be order preserving. Similarly, $\frac{r' - s'}{r - s} < 0$ if and only if $r' - s' > 0$ which is true if and only if the coordinate change is order reversing.

2. The coordinate change f determines an equation of the form $f(x) = x' = ax + b$. From $r' = ar + b$, $s' = as + b$. We find $a = \frac{r' - s'}{r - s}$, $b = \frac{rs' - r's}{r - s}$

   (a) f includes a contraction if and only if $0 < a < 1$ which is the condition $0 < \frac{r' - s'}{r - s} < 1$.

   (b) f includes a contraction and reflection if and only if $-1 < a < 0$ which is the condition $-1 < \frac{r' - s'}{r - s} < 0$.

   (c) f includes an expansion if and only if $a > 1$ which is the condition $\frac{r' - s'}{r - s} > 1$.

   (d) f includes an expansion and reflection if and only if $a < -1$ which is $\frac{r' - s'}{r - s} < -1$.

3. The coordinate change f determines an equation of the form $f(x) = ax + b$. From $p' = ap + b$, $q' = aq + b$ we find $a = \frac{p' - q'}{p - q}$, $b = \frac{pq' - p'q}{p - q}$.

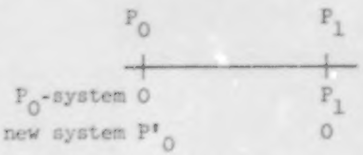
   (a) f includes a translation if and only if $a = 1$ which is the condition $\frac{p' - q'}{p - q} = 1$.

   (b) f includes a reflection if and only if $a = -1$ which is the condition $\frac{p' - q'}{p - q} = -1$.

4. We wish to show that the intrinsic coordinate systems are identical to the coordinate systems whose defining functions have the form $x' = x + b$ or $x' = -x + b$ with b any real number.

Pick one intrinsic coordinate system, call its origin P_0 and refer to it as the P_0-system.

Consider any other intrinsic coordinate system (one having the same unit length) with origin P_1 and the same positive direction.


P_0
P_1
P_0-system 0
P_1
new system P'_0
0

$x < y$ if and only if X is left of Y

$x' < y'$ if and only if X is left of Y

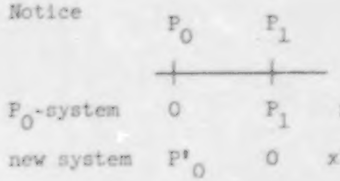
So $\vec{d}(P_0, P_1) = 0 - P'_0 = P_1 - 0$ since unit of measure is the same.

Solving $P_0' = a \cdot 0 + b$ and $0 = a \cdot P_1 + b$ we get $x' = x + (-P_1)$.

So this (intrinsic) coordinate system has defining function of the form $x' = x + b$ relative to the P_0-system. Conversely for any equation $x' = x + b$ we can find the intrinsic coordinate system whose origin has P_0 coordinate $(-b)$ and the P_0 positive direction.

Similarly we establish an identity between coordinate systems with positive sense opposite to that of the P_0-system and systems with defining functions $x' = -x + b$.

Notice

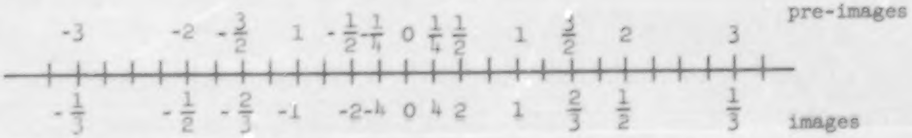

P_0
P_1
P_0-system
0
P_1
new system
P'_0
0

$x < y$ if and only if X is left of Y

$x' < y'$ if and only if X is right of Y

$\vec{d}(P_0, P_1)$ is $p_1 - 0$ in P_0-system, but $p'_0 - 0$ in system with opposite positive sense.

5.

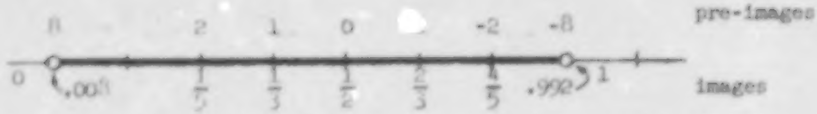

pre-images
-3
-2
$-\frac{3}{2}$
1
$-\frac{1}{2}$
$-\frac{1}{4}$
0
$\frac{1}{4}$
$\frac{1}{2}$
1
$\frac{3}{2}$
2
3
$-\frac{1}{3}$
$-\frac{1}{2}$
$-\frac{2}{3}$
-1
-2
-4
0
4
2
1
$\frac{2}{3}$
$\frac{1}{2}$
$\frac{1}{3}$
images

6. (a) Domain of $F(G(H))$ = domain of $H = \{w : w \text{ is real}\}$
range of $F(G(H)) = \{z : 0 < z < 1\}$

Transformation $F(G(R))$ is into the line, not onto.

It is one-to-one.

(b)



(c) The cardinality of the interior of a segment is the same as the cardinality of the line.

7. (a) Domain $D(E(F)) = \{w : w \text{ is real}\}$

Range $D(E(F)) = \{z : 0 < z < 1\}$

$D(E(F))$ maps the reals into but not onto the reals.

It is one-to-one.

(b) The cardinality of R is infinite.

8. Let the coordinate change be given by $x' = ax + b$.

Then $\frac{p' - q'}{r' - s'} = \frac{(ap + b) - (aq + b)}{(ar + b) - (as + b)} = \frac{a(p - q)}{a(r - s)} \quad \frac{(b - b)}{(b - b)} \quad \frac{p - q}{r - s}$

The operations are justified since $r \neq s$ and $a \neq c$ so tha

$r - s \neq 0$ and $\frac{a}{a} = 1$.

9. $x = \frac{11}{2}$

This may be obtained from the change of coordinate formula, or, using Problem 8, from ratio of directed distances (letting $A = P$, $B = R = Q$, $C = S$).

10. $x' = x(\frac{b' - a'}{b - a}) + (\frac{a'b - ab'}{b - a})$

11. Let f be a linear transformation of the line into itself such that for two distinct points X and Y, $f(X) = X$ and $f(Y) = Y$. We wish to show that for all points Z, $f(Z) = Z$.

$f(X) = X$ and $f(Y) = Y$ yield coordinate equations

$$x = ax + b \quad \text{and} \quad y = ay + b$$

which implies $a = 1$ and $b = 0$. So for any point Z with coordinate Z, $f(Z)$ has coordinate

$$z' = 1 \cdot z + 0 = Z.$$

So f keeps all points fixed.

Supplement D

(Supplement to Chapters 2,3,8)

POINTS, LINES, AND PLANES

In this chapter the student will face many problems arising from the relative positions of points, lines, and planes in space. Among these are the measurements of angles and distances, matters of parallelism and perpendicularity, and questions of incidence and separation.

Various schemes and devices are suggested as being appropriate in certain cases, but in the last analysis we believe that a student should not be told too much. He has many tools; therefore, he should be encouraged to find his own solution for any given situation.

Here is where a student begins to need some facility with determinants. There is help in Appendix A.

If the equation of a line is written in the form $ax + by + c = 0$, then the equations

$$\begin{aligned} ax_1 + by_1 + c &= 0 \\ ax_2 + by_2 + c &= 0 \\ ax_3 + by_3 + c &= 0 \end{aligned}$$

may be considered a system of 3 linear homogeneous equations in the 3 unknowns a, b, c. Equation (3) in the student's text is the necessary and sufficient condition that there are non-trivial solutions of the system.

Exercises D-2

1. (a) collinear (b) $k = 46.5$ (c) $|bc - ad|$ (d) collinear

2. ac; -ac; ac; -ac; yes; no. The direction of traverse of the triangle affects the sign (positive for counter-clockwise, negative for clockwise); the vertex at which one starts does not.

3. Consider the triangle with vertices $P_i = (x_i, y_i)$, $i = 1, 2, 3$. We know that the area is

$$K = \frac{1}{2}\left\|\begin{matrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{matrix}\right\| \qquad \text{i.e. absolute value of determinant}$$

$$= \frac{1}{2}\left|x_1(y_2 - y_3) - x_2(y_1 - y_3) + x_3(y_1 - y_2)\right|$$

$$= \frac{1}{2}\left|x_1y_2 - x_1y_3 - x_2y_1 + x_2y_3 + x_3y_1 - x_3y_2\right|$$

$$= \frac{1}{2}\left|(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3)\right|$$

$$= \frac{1}{2}\left(\left|\begin{matrix} x_1 & y_1 \\ x_2 & y_2 \end{matrix}\right| + \left|\begin{matrix} x_2 & y_2 \\ x_3 & y_3 \end{matrix}\right| + \left|\begin{matrix} x_3 & y_3 \\ x_1 & y_1 \end{matrix}\right|\right)$$

4. $18\frac{1}{2}$

5. (a) $\begin{vmatrix} -2 & 1 & 1 \\ 2 & -2 & 1 \\ 6 & -5 & 1 \end{vmatrix} = -2(3) - 2(6) + 6(3) = 0$

(b) $\vec{B} - \vec{A} = [4, -3]$ $\qquad \vec{C} - \vec{A} = [8, -6]$

Hence $\vec{B} - \vec{A} = \frac{1}{2}(\vec{C} - \vec{A})$

But $\overleftrightarrow{AB}$ is parallel to the line of $\vec{B} - \vec{A}$, and

$\overleftrightarrow{AC}$ is parallel to the line of $\vec{C} - \vec{A}$ which is the line of $\vec{B} - \vec{A}$.

So $\overleftrightarrow{AB}$ coincides with $\overleftrightarrow{AC}$.

(c) $d(A,B) = 5$, $d(B,C) = 5$, $d(A,C) = 10$

By the triangle inequality, this implies B lies on $\overleftrightarrow{AC}$.

If lines L_1 , L_2 , L_3 meet in a point (x_1,y_1) , then

$$a_1x_1 + b_1y_1 + c_1 = 0$$
$$a_2x_1 + b_2y_1 + c_2 = 0$$
$$a_3x_1 + b_3y_1 + c_3 = 0 \ .$$

This system of three linear equations in the two unknowns (x_1,y_1) has a common solution only if the determinant of the coefficients is zero; this condition is Equation (3) in the student's text.

It might be worthwhile to place considerable emphasis on the idea of families. This concept will appear later in connection with curves in the plane and in space.

Exercises D-3

1. (a) No (b) Yes, $(\frac{1}{2},\frac{5}{2})$ (c) No, (the lines are parallel)

2. (a) 4

   (b) $k^3 + 4k - 16 = (k - 2)(k^2 + 2k + 8) = 0$; real value, $k = 2$.

3. General form, $3x - 2y + 5 + n(x + 4y - 1) = 0$

   (a) $21x - 28y + 43 = 0$
   (b) $14x + 21y + 6 = 0$
   (c) $4x + 9y = 0$
   (d) $5x - 22y + 19 = 0$
   (e) $x - 3y + 3 = 0$

4. $9x - 3y + 8 = 0$

5. This exercise may be done in a variety of ways. If students use the methods in this section, some of the following may be useful in checking their work.

   (a) Centroid, $(\frac{a + c}{3},\frac{b}{3})$

   (b) Orthocenter, $(0,-\frac{ac}{b})$

   (c) Circumcenter, $(\frac{a + c}{2} , \frac{b^2 + ac}{2b})$

(d) Evaluate determinant in (3) of text by factoring out $\frac{a + c}{6b}$ from C_1 , $\frac{1}{6b}$ from C_2 , multiplying elements of R_2 by $-\frac{3}{2}$ and adding to elements of R_3 .

$$\begin{vmatrix} 0 & \frac{-ac}{b} & 1 \\ \frac{a + c}{3} & \frac{b}{3} & 1 \\ \frac{a + c}{2} & \frac{b^2 + ac}{2b} & 1 \end{vmatrix} = \frac{a + c}{6b \cdot 6b} \begin{vmatrix} 0 & -6ac & 1 \\ 2 & 2b^2 & 1 \\ 3 & 3b^2 + 3ac & 1 \end{vmatrix}$$

$$= \frac{a + c}{36b^2} \begin{vmatrix} 0 & -6ac & 1 \\ 2 & 2b^2 & 1 \\ 0 & 3ac & -\frac{1}{2} \end{vmatrix}$$

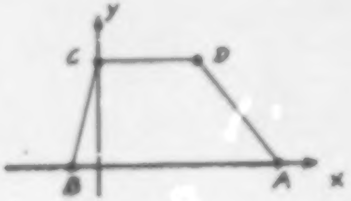
$$= \frac{a + c}{36b^2}(-2)(3ac - 3ac)$$

$$= 0$$

(e) Yes, because by appropriate choice of coordinates any triangle can have vertices with the coordinates given for A , B , C .

6. Consider trapezoid ABCD and choose coordinate system so that $A = (a,0)$, $B = (b,0)$, $C = (0,c)$, $D = (d,c)$. The diagonals are $cx + ay - ac = 0$, $cx + (b - d)y - bc = 0$. Joining midpoints of bases is the line $2cx + (a + b - d)y - (a + b)c = 0$

$$\begin{vmatrix} c & a & -ac \\ c & b - d & -bc \\ 2c & a + b - d & -(a + b)c \end{vmatrix} = 0$$



The subject matter of this course can be grouped and developed in various ways. Although we have used some of the contents of this section in earlier sections, we now consider, in a more systematic way, the general topic of intersections and parallelisms.

We make extensive use of determinants, with which we assume some reasonable familiarity. An appendix presents a brief treatment of the topic, which was considered too algebraic to be part of the text. Matrices also, would have facilitated our development, particularly the concept of the rank of a matrix, and an augmented matrix; but these ideas were considered to be too far afield from our central theme, and so do not appear, even in an appendix. Teachers and interested students are referred to the SMSG text on Matrix Algebra, or to any of the recent elementary texts on matrices. We recommend strongly that students be encouraged to gain some competence in those aspects of matrix algebra which apply to the present content, and perhaps prepare oral or written reports on these applications.

Authors, as well as students and teachers, are not pleased with pages that seem overloaded with letters and subscripts. However, in three dimensions, equations of lines and planes do require many symbols. We chose to use fewer letters with different subscripts, rather than many different letters, because we felt that, with a bit of effort, the patterns of relationships could be more easily seen. Students should be encouraged to see these patterns, and to try to extend them to corresponding situations in higher dimensions, where subscripts become more significantly necessary. We have avoided here, and generally throughout the text, the use of Σ notation. If students have the proper background and ability, they might be encouraged to state, as far as possible, the results of this section that could be generalized to n dimensions, using whatever symbolism they think most appropriate.

Solutions to Exercises D-4

1. (a) parallel
   (b) skew
   (c) skew
   (d) skew
   (e) skew
   (f) skew

2. (a) $\begin{cases} x = 1 + 3t \\ y = 2 - t \\ z = 3 - 2t \end{cases}$

(b) $\begin{cases} x = 1 - 6t \\ y = 2 + 2t \\ z = 3 + 4t \end{cases}$

(c) $\begin{cases} x = 1 + 3t \\ y = 2 - 2t \\ z = 3 - 8t \end{cases}$

(d) $\begin{cases} x = 1 - 3t \\ y = 2 + 4t \\ z = 3 - 6t \end{cases}$

3. (a) $M_1 : 4x + 18y - 3z - 34 = 0$

$M_2 : 4x + 18y - 3z - 69 = 0$

(b) $M_1 : 14x + 24y + 9z + 69 = 0$

$M_2 : 14x + 24y + 9z - 35 = 0$

4. (a) $4x + 18y - 3z - 34 = 0$

(b) $14x + 24y + 9z - 35 = 0$ Note $L_1 \parallel L_2$

5. (a) $2x - 8y + 7z = 0$

(b) $11x + 9y + 12z = 0$

(c) $22x + y + 8z = 0$

(d) $3y + 2z = 0$

6. (a) L_1 goes over L_4

(b) L_2 goes over L_3

(c) L_2 goes under L_4

(d) L_3 goes under L_4

7. If L_A goes over L_B and L_B goes over L_C, then it is sometimes true that L_A goes over L_C.

8. It is false that if L_A and L_B are distinct, then L_A goes over L_B or L_B goes over L_A. Consider the lines $L_A : x = 1$, $L_B : x = 2$. It is never the case that P_1 on L_A and P_2 on L_B have the same x-coordinate, hence, one criterion is never met.

9. (a) $[1,0,2] + t[5,11,7] = [x,y,z]$

(b) $[0,-11,-17] + t[1,7,7] = [x,y,z]$

(c) $[1,-1,0] + t[5,8,1] = [x,y,z]$

(d) $[3,2,4] + t[7,1,5] = [x,y,z]$

(e) $[1,-3,1] + t[5,2,4] = [x,y,z]$

(f) $[-5,-1,-6] + t[8,2,7] = [x,y,z]$

10. (a) $[\frac{11}{6}, \frac{11}{6}, \frac{19}{6}]$

(b) $[\frac{-2}{3}, \frac{-11}{3}, -\frac{1}{3}]$

(c) $(\frac{14}{9}, -\frac{1}{9}, \frac{1}{9})$

(d) $(\frac{58}{3}, \frac{13}{3}, \frac{47}{3})$

11. (a) $3x - 2y + z = 0$
(b) $2x + y - 3z = 0$
(c) $x + 3y - 2z = 0$
(d) $-2x + y + 2z = 0$

12. (a) $[\frac{7}{3}, \frac{14}{9}, \frac{10}{9}]$ (c) $[\frac{-23}{13}, \frac{50}{13}, \frac{57}{13}]$

(b) $[\frac{27}{11}, \frac{-53}{11}, \frac{15}{11}]$ (d) $[\frac{11}{2}, -4, 6]$

13. $L_1 \begin{cases} x = a_1 + \ell_1 t \\ y = b_1 + m_1 t \end{cases}$ $L_2 \begin{cases} x = a_2 + \ell_2 t \\ y = b_2 + m_2 t \end{cases}$

L_1 and L_2 are coincident if and only if

$$\begin{vmatrix} \ell_1 & \ell_2 \\ m_1 & m_2 \end{vmatrix} = 0$$

and there exists an s_0 such that

$$\begin{vmatrix} a_1 - a_2 & \ell_2 s_0 \\ b_1 - b_2 & m_2 s_0 \end{vmatrix} = 0$$

Note: This is equivalent to the existence of a t_0 such that

$$\begin{vmatrix} a_2 - a_1 & \ell_1 t_0 \\ b_2 - b_1 & m_1 t_0 \end{vmatrix} = 0 .$$

L_1 and L_2 are parallel if and only if

$$\begin{vmatrix} \ell_1 & \ell_2 \\ m_1 & m_2 \end{vmatrix} = 0$$

and there is no s_0 such than

$$\begin{vmatrix} a_1 - a_2 & \ell_2 s_0 \\ b_1 - b_2 & m_2 s_0 \end{vmatrix} = 0 .$$

L_1 and L_2 intersect in a unique point if and only if

$$\begin{vmatrix} \ell_1 & \ell_2 \\ m_1 & m_2 \end{vmatrix} \neq 0$$

It is traditional to talk about the angle between two lines, but present standards of precision require that we take account of the fact that at least four angles are formed when two lines intersect. These angles can be distinguished in a diagram by various methods, but all of these methods must induce a sense along each of the lines. We indicate explicitly in the text that such a sensing must underly any method of distinguishing these angles analytically.

It is convenient to carry through the development in the text using the parametric forms of equations for lines. We leave to an exercise (Problem 16) at the end of this section the development of some of these ideas, using the usual general forms of the equations of these lines, in 2-space. Students should be encouraged here, as in other places in the text, to use the corrdinate system and method of representation that seems most natural, and to be prepared to show the equivalence of the results obtained in different ways.

It is not expected that any class complete all the exercises at the end of this section. We have supplied sufficient exercises to give some variety in assignments, testing, etc.

Solutions to Exercises D-5

1. (a) ~172° $\cos \theta = \frac{-7\sqrt{2}}{10} \sim 0.9898$

   (b) ~75° $\cos \theta = \frac{3\sqrt{130}}{130} \sim 0.263$

   (c) ~83° $\cos \theta = \frac{-\sqrt{65}}{65} \sim -0.124$

2. (a) $\begin{cases} x = 3 + 3t \\ y = 5 + t \end{cases}$ or $y - 5 = \frac{1}{3}x - \frac{1}{3}$

(b) $\begin{cases} x = 3 + 2t \\ y = 5 + t \end{cases}$ or $y - 5 = \frac{1}{2}x - \frac{3}{2}$

(c) $\begin{cases} x = 3 - 2t \\ y = 5 + 3t \end{cases}$ or $y = \frac{-3}{2}x + \frac{19}{2}$

3. Lines $L_1 : y + 3x - 11 = 0$ direction pairs $\hat{L}_1 = [-1,3]$

$L_2 : y + 2x - 5 = 0$ $\hat{L}_2 = [-1,2]$

Bisectors $B_1 : (3 - 2\sqrt{2})x + (1 - \sqrt{2})y - 11 + 5\sqrt{2} = 0$ $\hat{B}_1 = [1-\sqrt{2},-3+2\sqrt{2}]$

$B_2 : (3 + 2\sqrt{2})x + (1 + \sqrt{2})y - 11 - 5\sqrt{2} = 0$ $\hat{B}_2 = [-1-\sqrt{2},3+2\sqrt{2}]$

Let θ be one angle determined by L_1 and B_2

Φ be one angle determined by L_2 and B_2

Since $\hat{L}_1$, $\hat{L}_2$ and $\hat{B}_2$ are in the same quadrant we can be sure that $\cos\theta = \cos\Phi$ implies that $\angle\theta \cong \angle\Phi$.

$$\cos\theta = \frac{\hat{B}_2 \cdot \hat{L}_1}{|\hat{B}_2||\hat{L}_1|} = \frac{10 + 7\sqrt{2}}{\left(\sqrt{20 + 14\sqrt{2}}\right)\sqrt{10}}$$

$$\cos\Phi = \frac{\hat{B}_2 \cdot \hat{L}_2}{|\hat{B}_2||\hat{L}_2|} = \frac{7 + 5\sqrt{2}}{\left(\sqrt{20 + 14\sqrt{2}}\right)\sqrt{5}} = \frac{10 + 7\sqrt{2}}{\left(\sqrt{20 + 14\sqrt{2}}\right)\sqrt{10}}$$

This can also be checked by noticing that $\cos\theta$ is the cosine of half the angle between $\hat{L}_1$ and $\hat{L}_2$.

4. (a) $P_1 = [\frac{11}{4}, 3]$ $P_2 = [\frac{-40}{11}, \frac{43}{11}]$ $P_3 = [6,-7]$

(b) Alt. from $P_1 = [\frac{11}{4}, 3] + t[3,1]$ line through $P_1 \perp L_1$

Alt. from $P_2 = [\frac{-40}{11}, \frac{43}{11}] + t[2,1]$ line through $P_2 \perp L_2$

Alt. from $P_3 = [6,-7] + t[-2,3]$ line through $P_3 \perp L_3$

5. The lines are parallel. Therefore, $\theta = 0^\circ$.

6. (a) $\arccos \frac{2}{\sqrt{154}} \approx \arccos 0.161 \approx 80.5^\circ$ and 99.5°

(b) $\arccos (\frac{-11}{14}) \approx 180^\circ - \arccos (0.786) \approx 141.7^\circ$ and 38.3°

(c) $\arccos (\frac{-8}{\sqrt{.54}}) \approx 180^\circ - \arccos (0.654) \approx 130^\circ$ and 50°

$$\left.\begin{array}{ll} \text{7. (a)} & [x,y,z] = [1,2,3] + t[a\ ,\ 3a - 2c\ ,\ c] \\ \text{(b)} & [x,y,z] = [1,2,3] + t[a\ ,\ a + 3c\ ,\ c] \\ \text{(c)} & [x,y,z] = [1,2,3] + t[a\ ,\ 3c - 2a\ ,\ c] \end{array}\right\} \text{for any } a \text{ and } c \text{ not both zero.}$$

8. (a) $N_1 : [x,y,z] = t[0,3,1]$

(b) $N_2 : [x,y,z] = t[1,1,1]$

(c) $N_3 : [x,y,z] = t[5,11,2]$

9. (a) $-3x + y + 2z - 10 = 0$
(b) $x - y + 3z - 19 = 0$
(c) $2x + y - 3z + 10 = 0$

10. (a) $5x + 11y + 2z - 51 = 0$
(b) $x + y + z - 9 = 0$
(c) $5x + 11y + 2z - 53 = 0$
(d) $3y + z - 14 = 0$
(e) $x + y + z - 7 = 0$
(f) $3y + z - 10 = 0$

11. (a) 86° and 94°

(b) 69° and 111°

(c) 60° and 120°

12. (a) $7x - y + 11z - 55 = 0$
(b) $x + 3y + 0z - 11 = 0$
(c) $3x - 12y + 7z + 2 = 0$
(d) $-8x + 7y + 5z - 62 = 0$
(e) $x + 7y + 2z - 35 = 0$
(f) $3x + 0y - z - 7 = 0$
(g) $2x - y + z - 4 = 0$
(h) $x + 13y + 5z - 47 = 0$
(i) $3x - 3y + z - 1 = 0$

13. (a) $5x - 7y - 11z = 0$
(b) $11x - 7y + z = 0$
(c) $x + y - z = 0$

14. (a) 21°
(b) 25.3°
(c) 4°
(d) 29.2°
(e) 53.6°
(f) 40.4°
(g) 45.6°
(h) 4°
(i) 21°

15.

	with x-axis	y-axis	z-axis
(a)	32.3°	53.2°	15.5°
(b)	53.2°	15.5°	32.3°
(c)	15.5°	32.3°	53.2°

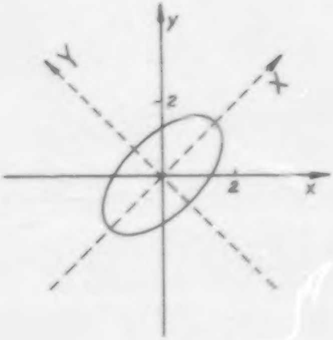
16. $\text{Cos} = \dfrac{a_1a_2 + b_1b_2}{\sqrt{a_1^2 + b_1^2}\ \sqrt{a_2^2 + b_2^2}}$

Supplement to Chapter 7

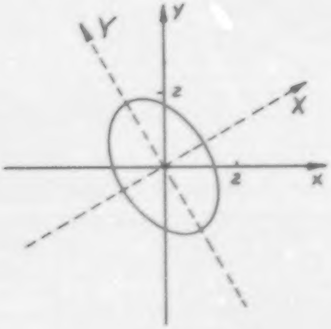
Exercises S7-6

1. (a) 27°
   (b) 60°
   (c) 22.5°
   (d) 36°
   (e) 30°
   (f) 63°

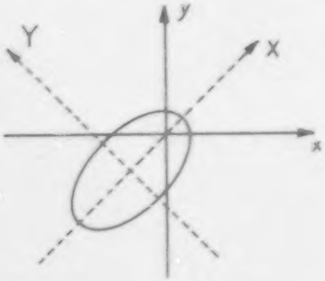
2. (a) $X^2 + 4Y^2 = 4$
   rotation through 45°
   ellipse



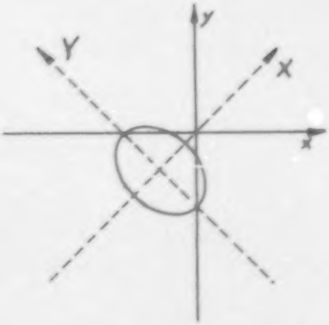
(c) $2X^2 + Y^2 = 4$
rotation through 30°
ellipse



(b) $X^2 + 4Y^2 = 4$
rotate 45°
translate $X = x + \sqrt{2}$
ellipse



(d) $2X^2 + Y^2 = 1$
rotate $\theta = 45$
translate $X = x + \sqrt{2}$
ellipse



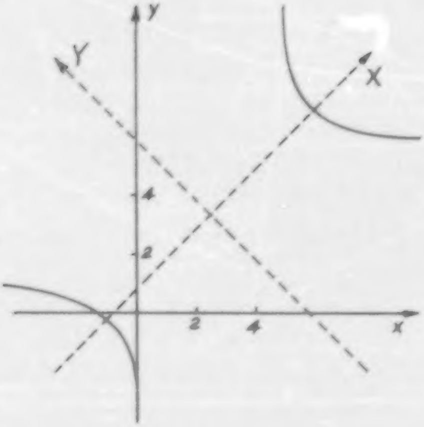
(e) $4X^2 - 8Y^2 = 99$

rotate 45°

translate $X = x - 3\sqrt{2}$,

$Y = y - \frac{\sqrt{2}}{4}$

hyperbola



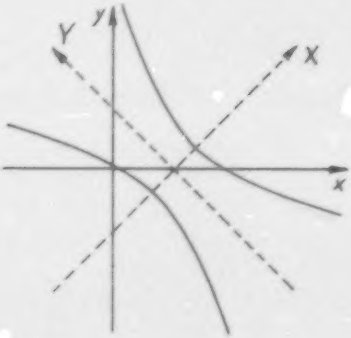
(f) $4X^2 - Y^2 = 4$

rotate arccos $\frac{4}{5}$

translate $X = x - \frac{8}{5}$,

$Y = y + \frac{6}{5}$

hyperbola

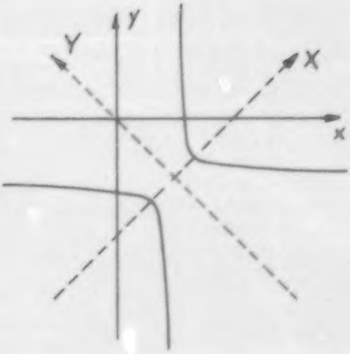


(g) $X^2 - Y^2 = 1$

rotate 45°

translate $X = x$, $Y = y + 2\sqrt{2}$

hyperbola

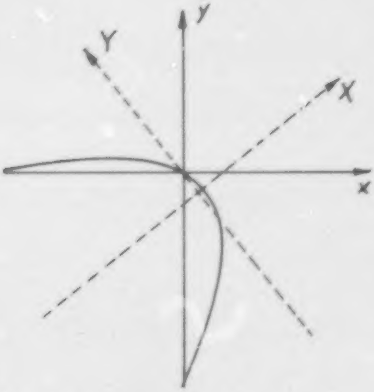


(h) $y^2 = -6x$

rotate arccos $\frac{4}{5}$

translate $X = x - \frac{1}{6}$, $Y = y + 1$

parabola



Exercises S7-7a

1. Given that $x' = x + h$

   and $y' = y + k$

   and $4x^2 + y^2 - 8x + 4y + 4 = 0$

   Find h and k such that the first-degree terms will be eliminated.

$$4x^2 + y^2 - 8x + 4y + 4 = 0 \qquad (1)$$

$$x = x' - h$$

$$y = y' - k$$

Substituting in (1) and grouping terms, we find that the transformed equation is

$$4x'^2 + y'^2 + (-8h - 8)x' + (-2k + 4)y' + (4h^2 + k^2 + 8h - 4k + 4) = 0$$

Solving simultaneously

$$-8h - 8 = 0 \qquad h = -1$$

$$-2k + 4 = 0 \qquad k = 2$$

The transformed equation becomes

$$4x'^2 + y'^2 = 4$$

$$F' = -4$$

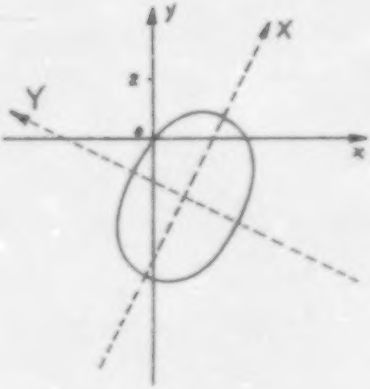
2. (a) $8x^2 - 4xy + 5y^2 - 24x + 24y = 0$

Translate to center $(1,-2)$

$8x'^2 - 4x'y' + 5y'^2 - 36 = 0$

Rotate through arctan 2

$$4X^2 - 9Y^2 = 36$$



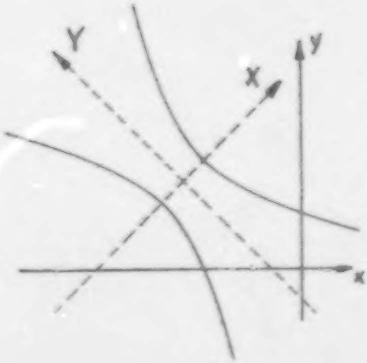
(b) $3x^2 + 10xy + 3y^2 - 6x + 22y - 53 = 0$

Translate to center $(-4,3)$

$3x'^2 + 10x'y' + 3y'^2 - 8 = 0$

Rotate through 45°

$$4X^2 - Y^2 = 4$$



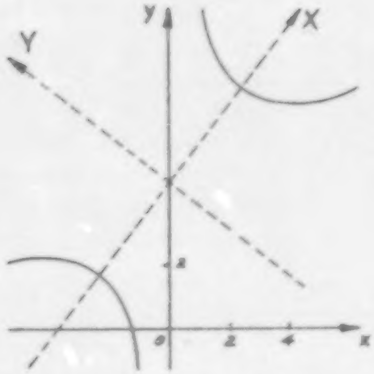
(c) $7x^2 - 24xy + 120x + 144 = 0$

Translate to center $(0,5)$

$$7x'^2 - 24x'y' + 144 = 0$$

Rotate through arctan $\frac{4}{3}$

$$9X^2 - 16Y^2 = 144$$



(d) $4x^2 - 8xy + 4y^2 - 9\sqrt{2}x \quad 7\sqrt{2}y + 14 = 0$

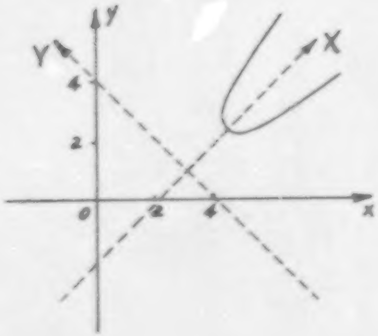
Translate to $(3,1)$

$$X = 2Y^2 + 2$$

Rotate through 45°

$$4y'^2 - 8y' - 2x' + 14 = 0$$

Parabola: $\delta = 0$



Exercises S7-7b

1. Center $(2,-5)$ Axes of symmetry $(y + 5) = \pm(x - 2)$

2. Center $(-\frac{11}{7}, -\frac{5}{7})$ Axes of symmetry $(y + \frac{5}{7}) = (\sqrt{17} - 4)(x + \frac{11}{7})$

$$(y + \frac{5}{7}) = -(\sqrt{17} + 4)(x + \frac{11}{7})$$

Exercises S7-8

1. (a) $0x^2 + 6xy + 0y^2 + 3x - 8y - 4 = 0$

$$\Delta = \begin{vmatrix} 0 & 6 & 3 \\ 6 & 0 & -8 \\ 3 & -8 & -8 \end{vmatrix} = -6(-24) - 6(24) = 0$$

Thus it is a degenerate conic: $(2y + 1)(3x - 4) = 0$

Lines: $2y + 1 = 0$, $3y - 4 = 0$

(b) $2x^2 + 8xy + 0y^2 - x + 4y - 1 = 0$

$$\Delta = \begin{vmatrix} 4 & 8 & -1 \\ 8 & 0 & 4 \\ -1 & 4 & -2 \end{vmatrix} = 4(-16) - 8(-12) - 32 = 0$$

Thus it is a degenerate conic: $(2x + 1)(x + 4y - 1) = 0$

Lines: $2x + 1 = 0$, $x + 4y - 1 = 0$

(c) $4x^2 - 5xy + 9y^2 - 1 = 0$

$$\Delta = \begin{vmatrix} 8 & -5 & 0 \\ -5 & 18 & 0 \\ 0 & 0 & -2 \end{vmatrix} = 8(-36) + 5(10) = -288 + 50 \neq 0$$

Thus it is not a degenerate conic.

(d) $2x^2 - 1xy - 6y^2 = 0$

$$\Delta = \begin{vmatrix} 4 & -1 & 0 \\ -1 & -12 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

So it is a degenerate conic: $(2x + 3)(x - 2y) = 0$

Lines: $2x + 3 = 0$, $x - 2y = 0$

2. Consider $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$
where $\Delta = 0$ and $\delta \neq 0$.

Case 1. Suppose the factors of the left member represent dependent linear equations. Then we could write the left member as
$(Mx + Ny + P)(kMx + kNy + kP) = 0$ where $k \neq 0$.
But then we get

$$kM^2x + 2kMNxy + kN^2y^2 + 2kMPx + 2kNPy + kP^2 = 0$$

$\delta = 4(kM^2)(kN^2) - (2kMN)^2 = 0$ which contradicts our hypothesis $\delta \neq 0$.

Case 2. Supposing the factors represent inconsistent equations, we get that

$$(Mx + Ny + P)(kMx + kNy + hP) = 0 \quad \text{for } k \neq 0,\ h \neq k.$$

But again this implies that $\delta = 0$ contrary to our hypothesis, $\delta \neq 0$.

3. Consider $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$
where

$$\Delta = \begin{vmatrix} 2A & B & D \\ B & 2C & E \\ D & E & 2F \end{vmatrix} = 2F \quad - E(2AE - BD) + D(BE - 2CD) = 0$$

and

$$\delta = \begin{vmatrix} 2A & B \\ B & 2C \end{vmatrix} = 0 .$$

Then $\qquad -2AE^2 + BDE + BDE - 2CD^2 = 0$

or $\qquad -2AE^2 + BDE = 2CD^2 - BDE = 0$.

Expression (5) is $(B^2 - 4AC)x^2 + 2(BE - 2CD)x + E^2 - 4CF$.

$\delta = 4AC - B^2 = 0$ makes the coefficient of x^2 vanish.
It remains to show that the coefficient of x is 0.

From $\Delta = 0$ and $B^2 = 4AC$ we get

$$0 = -AE^2 + BDE - CD^2 .$$

Multiply by $-4A$ and use $B^2 = 4AC$ to get

$$0 = 4A^2E^2 - 4ABDE + 4ACD^2$$

$$0 = 4(AE)^2 - 4(AE)(BD) + 4(BD)^2$$

$$0 = (2AE - BD)^2 .$$

Hence $BD - 2AE = 0$ which completes the proof.

Exercises 87-10

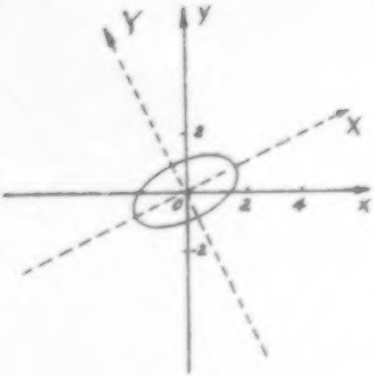
1. $8x^2 - 12xy + 17y^2 - 20 = 0$

$\delta = 400 \qquad \Delta = -16000$

Rotate through $\frac{1}{2}$ arctan $\frac{4}{3}$

$$X^2 + 4Y^2 = 4$$

ellipse



3. $5x^2 - 6xy + 5y^2 - 16x + 16y + 8 = 0$

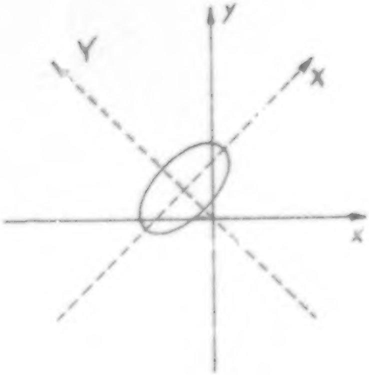
$\delta = 64 \qquad \Delta = -1024$

Translate $h = 1$, $k = -1$

Then rotate through 45°

$$X^2 + 4Y^2 = 4$$

ellipse



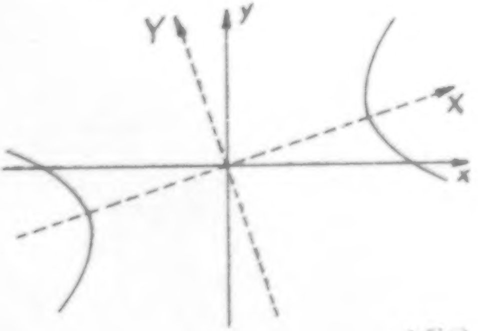
2. $3x^2 + 12xy - 13y^2 - 135 = 0$

$\delta = -300 \qquad \Delta = 81000$

Rotate through $\frac{1}{2}$ arctan $\frac{3}{4}$

$$X^2 - 3Y^2 = 27$$

hyperbola



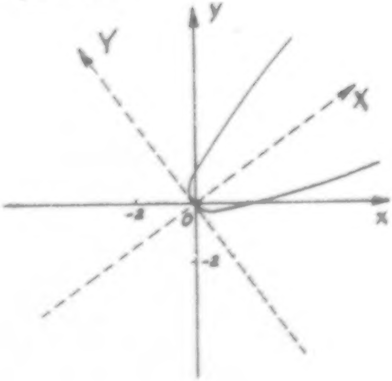
4. $9x^2 - 24xy + 16y^2 - 20x - 15y = 0$

$\delta = 0 \qquad \Delta = -8500$

Rotate through arccos $\frac{4}{5}$

$$Y^2 = X$$

parabola



5. $9x^2 - 24xy + 16y^2 + 60x - 80y + 100 = 0$

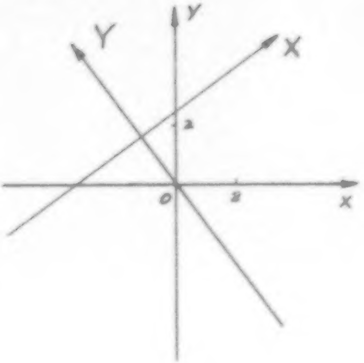
$\delta = 0 \quad \Delta = 0$

Rotate through arccos $\frac{4}{5}$

Translate $Y = y - 2$, $X = x$

$$Y = 0$$

coincident lines



6. $3x^2 + 10xy + 3y^2 + 16x + 16y + 24 = 0$

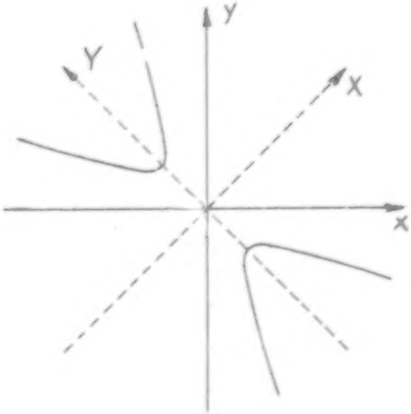
$\delta = -64 \quad \Delta = 512$

Rotate through 45°

Translate $Y = y$, $X = x + \sqrt{2}$

$$Y^2 - 4X^2 = 4$$

hyperbola



7. $5x^2 + 6xy + 5y^2 - 16x - 16y + 8 = 0$

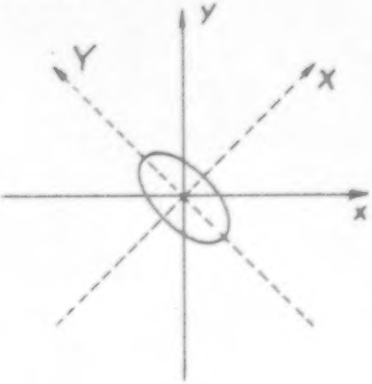
$\delta = 64 \quad \Delta = -1024$

Rotate through 45°

Translate $X = x - \sqrt{2}$, $Y = y$

$$4X^2 + Y^2 = 4$$

ellipse



8. $27x^2 - 48xy + 13y^2 - 12x + 44y - 77 = 0$

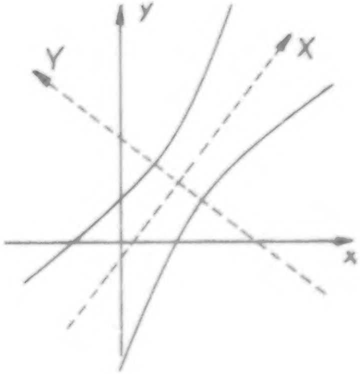
$\delta = -900 \quad \Delta = -196200$

Rotate through arccos $\frac{3}{5}$

Translate $X = x - \frac{14}{5}$, $Y = y + \frac{2}{5}$

$$9Y^2 - X^2 = 9$$

hyperbola



9. $12x^2 - 7xy - 12y^2 - 41x + 38y + 22 = 0$

$\delta = -625 \quad \Delta = 0$

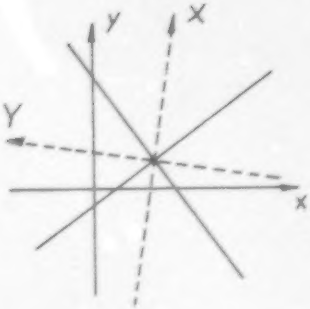
Rotate through arccos $\frac{1}{5\sqrt{2}}$

Translate $X = x - \frac{9}{5\sqrt{2}}$,

$Y = y + \frac{\text{[illegible]}}{5\sqrt{2}}$

$(X + Y)(X - Y) = 0$

Intersecting lines



11. $9x^2 - 24xy + 16y^2 + 90x - 120y + 200 = 0$

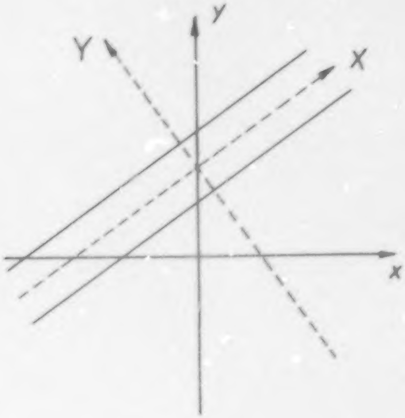
$\delta = 0 \quad \Delta = 0$

Rotate through arccos $\frac{4}{5}$

Translate $X = x$, $Y = y - 3$

$(Y - 1)(Y + 1) = 0$

Parallel lines



10. $13x^2 + 48xy + 27y^2 + 44x + 12y - 77 = 0$

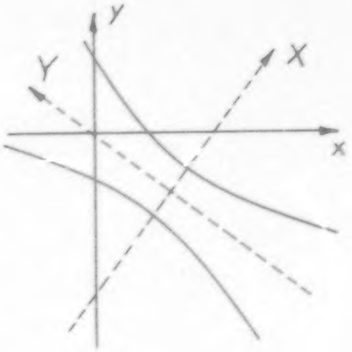
$\delta = -900 \quad \Delta = -196200$

Rotate arccos $\frac{3}{5}$

Translate $X = x + \frac{2}{5}$, $Y = y + \frac{14}{5}$

$9X^2 - Y^2 = 9$

hyperbola



12. $10xy + 4x - 15y - 6 = 0$

$\delta = -100 \quad \Delta = 0$

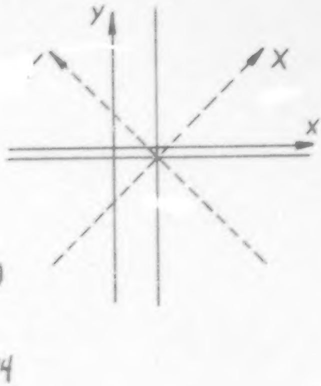
Rotate 45°

Translate $X = x - \frac{11\sqrt{2}}{20}$,

$Y = y + \frac{19\sqrt{2}}{20}$

$(X + Y)(X - Y) = 0$

intersecting lines



Supplement to Chapter 10

GEOMETRIC TRANSFORMATIONS

In a sense, this chapter can be thought of as a review of the early chapters. It is essentially a summary of the various treatments of transformations, but now they are observed from a more sophisticated point of view. The concepts of mappings and groups constitute the background for the discussion.

The writers would be interested in knowing how the teachers feel about including this type of material and also, if it is included, whether it should come earlier in the presentation--perhaps even near the front of the book.

Exercises S10-2

1. The reflection about the $x = 1$ line is $(x,y) \longrightarrow (x',y') = (-x+2,y)$.
The reflection about the $x = 4$ line is $(x',y') \longrightarrow (x'',y'') = (-x'+8,y')$.
Taking $x = 1$ then $x = 4$ we get

$$\boxed{x'' = x + 6 \, , \; y'' = y}$$

Taking $x = 4$ then $x = 1$ we get

$$\boxed{x'' = -x' + 2 = -(-x + 8)'' + 2 = x - 6} \quad \boxed{y'' = y}$$

So they don't commute.

2. Mapping of reflection about $x = h$

$$\boxed{(x,y) \longrightarrow (x',y') = (-x + 2h, y)}$$

Mapping of reflection about $y = k$

$$\boxed{(x,y) \longrightarrow (x',y') = (x, -y + 2k)}$$

3. Two successive reflections about horizontal lines:

$$(x,y) \longrightarrow (x',y') = (x, -y + 2k), \quad (x',y') \longrightarrow (x'',y'') = (x', -g' + 2n)$$

$$\begin{array}{l} x'' = x' = x \\ y'' = -y' + 2n = \end{array} \boxed{\begin{array}{r} x'' = x \\ y + 2(n - k) = y'' \end{array}}$$

Two successive reflections about vertical lines:

$$(x,y) \longrightarrow (x',y') = (-x + 2h, y), \quad (x',y') \longrightarrow (x'', y'') = (-x' + 2m, y')$$

$$\begin{array}{l} x'' = -x' + 2m = \\ y'' = y' = y \end{array} \boxed{\begin{array}{r} x + 2(m - h) = x'' \\ y'' = y \end{array}}$$

4. $(x,y) \longrightarrow (x',y') = (-x + 2h)y, \quad (x',y') \longrightarrow (x'',y'') = (x', -y' + 2k)$

$$\begin{array}{l} x'' = x' = \\ y'' = -y' + 2k = \end{array} \boxed{\begin{array}{r} -x + 2h = x'' \\ -y + 2k = y'' \end{array}}$$

5. The mappings in (3) will commute only if $k = n$ and $h = m$.

   The mappings in (4) will commute.

Exercises S10-3

1. Suppose they have the rotation

$$\phi'' = \phi + 2(\theta_2 - \theta_1)$$
$$r'' = r$$

Then rewrite

$$\phi'' = 2\theta_2 - (2\theta_1 - \phi)$$
$$r'' = r$$

Then let $r = r'$ and $2\theta_1 - \phi = \phi'$ and we have $\phi'' = 2\theta_2 - \phi'$, $r'' = r$.

Then we see that the rotation is the product of the line reflections

$$(r,\phi) \longrightarrow (r',\phi') = (r, 2\theta_1 - \phi) \text{ and}$$
$$(r',\phi') \longrightarrow (r'',\phi'') = (r', 2\theta_2 - \phi')$$

2. R_LR_M where $R_m : (r,\phi) \longrightarrow (r',\phi') = (r, 2\theta_2 - \phi)$

$R_L : (r',\phi') \longrightarrow (r'',\phi'') = (r', 2\theta_1 - \phi')$

$$\begin{array}{l} \phi'' = 2\theta_1' - \phi' = \\ r'' = r \end{array} \boxed{\begin{array}{r} \phi + 2(\phi_1 - \phi_2) = \phi'' \\ r = r'' \end{array}}$$

Exercises S10-4

1. $(x,y) \longrightarrow (x',y') = (ax + by, cx + dy)$ where $ad - bc \neq 0$

Now solve for x and y in terms of x' and y'.

Then $y = \dfrac{cx' - ay'}{bc - ad}$ and $x = \dfrac{dx' - by'}{ad - bc}$.

Now substitute these into the line $kx + \ell y + m = 0$ and we see that

$$kdx' - kby' + \ell cx' - \ell ay' + m = 0$$

or

$$(kd + \ell c)x' + (-kb - \ell a)y' + m = 0$$

which means that any transformation of the group in Theorem S10-3 will map a line into a line.

2. (a) $(x,y) \longrightarrow (2x,2y)$

$x' = 2x$, $y' = 2y$

$x'^2 + y'^2 = 4(x^2 + y^2)$ so the circle $x^2 + y^2 = 1$ maps into $x'^2 + y'^2 = 4$.

(b) $(x,y) \longrightarrow (2x,3y)$

$x' = 2x$, $y' = 3y$

$x^2 + y^2 = \frac{1}{4}x'^2 + \frac{1}{9}y'^2 = 1$ so the circle $x^2 + y^2 = 1$

maps into the ellipse $\frac{1}{4}x'^2 + \frac{1}{9}y'^2 = 1$

3. $(x,y) \longrightarrow (x',y') = (x + y, 2x + 2y)$

$x' = x + y$, $y' = 2x + 2y$

Consider the point a, $2a$ on $2x = y$, then $a = x + y$ and $2a = 2x + y$ so all points mapped into a point on $2x = y$ satisfy the equation $x + y - a = 0$. This is the equation of a line.

4. Show that the angle is preserved between two lines through the origin under $z \longrightarrow z' = kz$.

Let $z = r(\cos\theta + i\sin\theta)$, then let L_1 be $r(\cos\theta_1 + i\sin\theta_1)$ and L_2 be $r(\cos\theta_2 + i\sin\theta_2)$. Now the angle between L_2 and L_1 will simply be $|\theta_2 - \theta_1|$. Under the mapping $L_1 \longrightarrow L_1'$ where L_1' is $Kr(\cos\theta_1 + i\sin\theta_1)$ and $L_2 \longrightarrow L_2'$ where L_2' is $Kr(\cos\theta_2 + i\sin\theta_2)$. So we see the angle between L_1' and L_2' again equals $|\theta_2 - \theta_1|$. Therefore the angle is preserved.

5. Discuss $z - z' = \frac{1}{z}$

$$z = x + iy \; , \; \frac{1}{z} = z' = \frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2}$$

so $x' = \frac{x}{x^2 + y^2}$ and $y' = \frac{-y}{x^2 + y^2}$ in non-linear coordinates.

Then the circles $(x - \frac{1}{k})^2 + y^2 = \frac{1}{4k^2}$ are mapped onto $x' = k$ and the circles $x^2 + (y + \frac{1}{k})^2 = \frac{1}{4k^2}$ are mapped onto $y' = k$.

Also we have $x'^2 + y'^2 = \frac{x^2 + y^2}{(x^2 + y^2)^2} = \frac{1}{x^2 + y^2}$, hence the circles $x^2 + y^2 = r$ are mapped onto the circles $x'^2 + y'^2 = \frac{1}{r}$, in the z' plane.

6. (a) It is simplest to consider this problem in polar coordinates then the solution is $(r,\phi) \longrightarrow (r',\phi') = (\frac{1}{r},\phi')$ where the origin is defined to map onto the origin.

(b) A second form would be $(x,y) \longrightarrow (x',y') = (\frac{1}{x(1 + a^2)} \; , \; y)$ where $y = ax$ is the line involved. Again the origin would have to be defined as mapping onto the origin.

Exercises S10-5a

1. $R_x \ R_y$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

2. (a) Reflection about $y = x$

$$x' = y = 0 \cdot x + 1 \cdot y$$
$$y' = x = 1 \cdot x + 0 \cdot y$$
$$\boxed{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$$

(b) Reflection about $y = -x$

$$x' = -y = 0 \cdot x + -1 \cdot y$$
$$y' = -x = -1 \cdot x + 0 \cdot y$$
$$\boxed{\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}}$$

3. Reflection in $y = x$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

rotation $\frac{\pi}{2}$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

composition is

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \boxed{\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}}$$

4. $$\begin{pmatrix} \cos \theta_2 & - \sin \theta_2 \\ \sin \theta_2 & \cos \theta_2 \end{pmatrix} \cdot \begin{pmatrix} \cos \theta_1 & -\sin \theta_1 \\ \sin \theta_1 & \cos \theta_1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos \theta_2 \cos \theta_1 - \sin \theta_2 \sin \theta_1 \ , & -\sin \theta_1 \cos \theta_2 - \sin \theta_2 \cos \theta_1 \\ \cos \theta_1 \sin \theta_2 + \cos \theta_2 \sin \theta_1 \ , & \sin \theta_1 \sin \theta_2 + \cos \theta_1 \cos \theta_2 \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\theta_1 + \theta_2) & -\sin(\theta_1 + \theta_2) \\ \sin(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2) \end{pmatrix}$$

This mapping is the same as a mapping of a single rotation through $\theta_1 + \theta_2$ radians.

5. $$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \cdot \left[\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \cdot \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} \right] = k$$

$$K = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \cdot \begin{pmatrix} b_1c_1 + b_2c_2 & b_1c_2 + b_2c_4 \\ b_3c_1 + c_3b_4 & b_3c_2 + b_4c_4 \end{pmatrix}$$

$$K = \begin{pmatrix} a_1b_1c_1 + a_1b_2c_3 + a_2b_3c_1 + a_2c_3b_4 & a_1b_1c_2 + a_1b_2c_4 + a_2b_3c_2 + a_2b_4c_4 \\ a_3b_1c_1 + a_3b_2c_3 + a_4b_3c_1 + a_4c_3b_4 & a_3b_1c_2 + a_3b_2c_4 + a_4b_3c_2 + a_4b_4c_4 \end{pmatrix}$$

and

$$\left[\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \cdot \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \right] \cdot \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} = K'$$

$$K' = \begin{pmatrix} a_1b_1 + a_2b_3 & a_1b_2 + a_2b_4 \\ a_3b_1 + a_4b_3 & a_3b_2 + a_4b_4 \end{pmatrix} \cdot \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}$$

$$K' = \begin{matrix} a_1b_1c_1 + a_2b_3c_1 + a_1b_2c_3 + a_2b_4c_3 & a_1b_1c_2 + a_2b_3c_2 + a_1b_2c_4 + a_2b_4c_4 \\ a_3b_1c_1 + a_4b_3c_1 + a_3b_2c_3 + a_4b_4c_3 & a_3b_1c_2 + a_4b_3c_2 + a_3b_2c_4 + a_4b_4c_4 \end{matrix}$$

and so we see that $K = K'$ and matrix multiplication is associative.

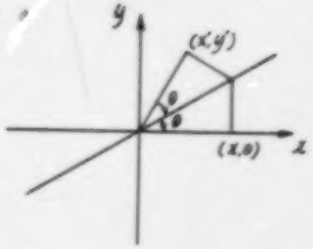
$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \cdot \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} = \begin{pmatrix} a_1b_1 + a_2b_3 & a_1b_2 + a_2b_4 \\ a_3b_1 + a_4b_3 & a_3b_2 + a_4b_4 \end{pmatrix} = L$$

$$\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \cdot \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} = \begin{pmatrix} b_1a_1 + b_2a_3 & b_2a_1 + b_4a_2 \\ b_3a_1 + b_4a_3 & b_3a_2 + b_4a_4 \end{pmatrix} = L'$$

and so we see that $L \neq L'$ hence matrix multiplication doesn't commute.

6. In polar coordinates

$r' = r$ and $\phi' = 2\theta - \phi$



$$x' = r\cos(2\theta - \phi) = r\cos\phi\cos 2\theta + r\sin\phi\sin 2\theta = x\cos 2\theta + y\sin 2\theta$$
$$y' = r\sin(2\theta - \phi) = r\sin 2\theta\cos\phi - r\cos 2\theta\sin\phi = x\sin 2\theta - y\cos 2\theta$$

hence the matrix is:

$$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$

When $\theta = 0$, we get $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ which was previously shown to be a reflection about the x-axis, when $\theta = \frac{\pi}{4}$ we get $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ which was previously shown to be a reflection in $y = x$, when $\theta = \frac{\pi}{2}$ we get $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ which is a reflection in the y-axis, when $\theta = \frac{3\pi}{4}$ we get $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ which is a reflection in the $y = -x$-axis.

7. $$\begin{pmatrix} \cos 2\theta_2 & \sin 2\theta_2 \\ \sin 2\theta_2 & -\cos 2\theta_2 \end{pmatrix} \cdot \begin{pmatrix} \cos 2\theta_1 & \sin 2\theta_1 \\ \sin 2\theta_1 & -\cos 2\theta_1 \end{pmatrix} =$$

$$\begin{pmatrix} \cos 2\theta_2 \cos 2\theta_1 + \sin 2\theta_2 \sin 2\theta_1 & \cos 2\theta_2 \sin 2\theta_1 - \cos 2\theta_1 \sin 2\theta_2 \\ \cos 2\theta_1 \sin 2\theta_2 - \cos 2\theta_2 \sin 2\theta_1 & + \sin 2\theta_1 \sin 2\theta_2 + \cos 2\theta_1 \cos 2\theta_2 \end{pmatrix}$$

$$= \begin{pmatrix} \cos 2(\theta_2 - \theta_1) & -\sin 2(\theta_2 - \theta_1) \\ \sin 2(\theta_2 - \theta_1) & \cos 2(\theta_2 - \theta_1) \end{pmatrix}$$

This is the matrix of a rotation where $\theta = 2(\theta_2 - \theta_1)$

Exercises S10-5b

1. $\begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix}$ or $\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$

By Problem 7 (S10-5a) we saw that the product of two matrices of the form $\begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix}$ is of the form $\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$.

By Problem 4 (S10-5a) we saw that the product of two matrices of the form $\begin{matrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & -\cos\alpha \end{matrix}$ is another matrix of the same form.

We see that the product $\begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix} \cdot \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & +\cos\alpha \end{pmatrix}$ is of the form $\begin{pmatrix} \cos\beta & \sin\beta \\ \sin\beta & -\cos\beta \end{pmatrix}$.

Finally $\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \cdot \begin{pmatrix} \cos\beta & +\sin\beta \\ \sin\beta & -\cos\beta \end{pmatrix}$ is of the form $\begin{pmatrix} \cos\alpha+\beta & \sin\alpha+\beta \\ \sin\alpha+\beta & -\cos\alpha+\beta \end{pmatrix}$.

Hence we see that the matrix multiplication is closed. From Problem 5 (S10-5a) we see that the multiplication obeys the associative law, and because $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$ is included in this set and it is the identity matrix, that this set forms a group.

2. $\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \cdot \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} = \begin{pmatrix} a_1b_1 + a_2b_3 & a_1b_2 + a_2b_4 \\ a_3b_1 + a_4b_3 & a_3b_2 + a_4b_4 \end{pmatrix}$

$$\begin{vmatrix} a_1b_1 + a_2b_3 & a_1b_2 + a_2b_4 \\ a_2b_1 + a_2b_3 & a_3b_2 + a_4b_4 \end{vmatrix} = (a_1b_1 + a_2b_3)(a_3b_2 + a_4b_4) - (a_3b_1 + a_4b_3)(a_1b_2 + a_2b_4)$$

$$= a_1b_1a_4b_4 + a_2a_3b_2b_3 - a_2a_3b_1b_4 - a_1a_4b_2b_3$$

$$= (a_1a_4 - a_2a_3)(b_1b_4 - b_2b_3)$$

$$= \begin{vmatrix} a_1 & a_2 \\ a_3 & a_4 \end{vmatrix} \cdot \begin{vmatrix} b_1 & b_2 \\ b_3 & b_4 \end{vmatrix}$$

3. The matrix $\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$ isn't an isometry as the vector $(0,1) \longrightarrow (2,1)$ and hence distance isn't preserved, yet the det = 1

4. The matrix must be of the form $\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$ or $\begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix}$ by Theorem 10-5.

$$\begin{vmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{vmatrix} = \cos^2\alpha + \sin^2\alpha = 1$$

$$\begin{vmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{vmatrix} = -\cos^2\alpha - \sin^2\alpha = -1$$

Hence the det of the matrix that represents an isometry is 1 or -1 .

5. If $\begin{vmatrix} a_1 & a_2 \\ a_3 & a_4 \end{vmatrix} = \pm 1$ then $a_1a_4 - a_2a_3 = \pm 1$; also, we have $a_1^2 = a_3^2 = 1$, $a_1^2 + a_2^2 = 1$, $a_3^2 + a_4^2 = 1$ and $a_2^2 + a_4^2 = 1$. Now, if the sum of two squares = 1, the numbers can be written as sin and cos of some angle θ . Hence we have $a_1 = \pm \sin\alpha$ or $\pm \sin\alpha$, $a_2 = \pm \cos\alpha$ or $\pm \sin\alpha$, $a_3 = \pm \sin\alpha$ or $\pm \cos\alpha$, $a_4 = \pm \cos\alpha$ or $\pm \sin\alpha$. Now, from these, we obviously get matrices that belong to S out we get other as well: $a_3 = \pm \sin\alpha$ or $\pm \cos\alpha$, $a_4 = \pm \cos\alpha$ or $\pm \sin\alpha$. Now from these we obviously get matrices that belong to S but we get others as well: $\begin{pmatrix} -\cos\alpha & \sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$, $\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$, $\begin{pmatrix} \sin\alpha & \cos\alpha \\ -\cos\alpha & \sin\alpha \end{pmatrix}$, $\begin{pmatrix} \sin\alpha & \cos\alpha \\ \cos\alpha & -\sin\alpha \end{pmatrix}$ $\begin{pmatrix} \sin\alpha & -\cos\alpha \\ \cos\alpha & \sin\alpha \end{pmatrix}$, and $\begin{pmatrix} -\sin\alpha & \cos\alpha \\ \cos\alpha & \sin\alpha \end{pmatrix}$. All of these cases can be reduced to members of S by letting $\alpha = -\beta$, $\alpha = \beta + \frac{\pi}{2}$ or $\alpha = \beta + \pi$. Hence, these conditions are enough to make the matrix belong to S .

Exercises S10-6

1. Answers given in text

2. Answers given in text

3. I-1 Reflection in x-y plane
I-2 Reflection in y-z plane
I-3 Reflection in x-z plane
I-4 Identity
I-5 Reflection in plane through x-axis with 45° to y-axis
I-6 Reflection in plane through y-axis with 45° angle to z-axis
I-7 Reflection in plane through z-axis with 45° angle to x-axis
I-8 Reflection in plane through x-axis with 135° angle to y-axis
I-9 Reflection in plane through y-axis with 135° angle to z-axis
I-10 Reflection in plane through z-axis with 135° angle to x-axis